

One-parameter convolution semigroups of rapidly decreasing distributions

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Abstract

Let $M_{m \times m}$ denote the set of $m \times m$ matrices with complex entries, and let $\mathcal{G}(\partial_1, \dots, \partial_n)$ be an $m \times m$ matrix whose entries are partial differential operators on \mathbb{R}^n with constant complex coefficients. It is proved that $\mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta$ is the generating distribution of a smooth one-parameter convolution semigroup of $M_{m \times m}$ -valued rapidly decreasing distributions on \mathbb{R}^n if and only if

$$\sup_{(\xi_1, \dots, \xi_n) \in \mathbb{R}^n} \operatorname{Re} \sigma(\mathcal{G}(i\xi_1, \dots, i\xi_n)) < \infty.$$

Applications to systems of partial differential operators with constant coefficients are considered.

Introduction

One-parameter semigroups in the convolution algebra of rapidly decreasing distributions

Let $M_{m \times m}$ be the set of $m \times m$ matrices with complex entries, and $\mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ the convolution algebra of $M_{m \times m}$ -valued distributions on \mathbb{R}^n rapidly decreasing in the sense of L. Schwartz. The Fourier transformation \mathcal{F} is an isomorphism of $\mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ onto the algebra $\mathcal{O}_M(\mathbb{R}^n; M_{m \times m})$ of $M_{m \times m}$ -valued infinitely differentiable slowly increasing functions on \mathbb{R}^n . We prove

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that $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ is the generating distribution of a one-parameter infinitely differentiable convolution semigroup $(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ if and only if

$$\max\{\operatorname{Re} \lambda : \lambda \in \sigma((\mathcal{F}G)(\xi))\} = O(\log |\xi|) \quad \text{as } |\xi| \rightarrow \infty. \quad (\text{i})$$

In the above, σ denotes the spectrum of a square matrix.

If $G = \mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta$ where δ is the Dirac distribution on \mathbb{R}^n , $\partial_1, \dots, \partial_n$ denote the first order partial derivatives with respect to the coordinates of \mathbb{R}^n , and $\mathcal{G}(\partial_1, \dots, \partial_n)$ is an $m \times m$ matrix whose entries are scalar partial differential operators (PDOs) with constant coefficients, then $(\mathcal{F}G)(\xi) = \mathcal{G}(i\xi)$ for every $\xi \in \mathbb{R}^n$, and condition (i) takes the form

$$\max\{\operatorname{Re} \lambda : \lambda \in \sigma(\mathcal{G}(i\xi))\} = O(\log |\xi|) \quad \text{as } |\xi| \rightarrow \infty. \quad (\text{i})'$$

Thanks to the fact that $\det(\lambda \mathbb{1}_{m \times m} - \mathcal{G}(\zeta_1, \dots, \zeta_n))$ is a polynomial, L. Gårding was able to prove the conjecture of I. G. Petrovskiĭ that (i)' is equivalent to the condition

$$\sup\{\operatorname{Re} \lambda : \lambda \in \sigma(\mathcal{G}(i\xi)), \xi \in \mathbb{R}^n\} < \infty. \quad (\text{ii})$$

Application to the Cauchy problem for partial differential equations with constant coefficients

Suppose that $\mathcal{G}(\partial_1, \dots, \partial_n)$ satisfies (ii), and $(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ is the infinitely differentiable convolution semigroup with generating distribution $\mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta$. Suppose moreover that

E is a sequentially complete l.c.v.s. continuously imbedded in $\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$ such that $(S_t *)E \subset E$ for every $t \in [0, \infty[$, and the mapping $[0, \infty[\times E \ni (t, u) \mapsto S_t * u \in E$ is separately continuous. (iii)

Then $((S_t *)|_E)_{t \geq 0} \in L(E; E)$ is a one-parameter operator semigroup of class (C_0) whose infinitesimal generator \mathcal{G}_E satisfies the equalities

$$\begin{aligned} D(\mathcal{G}_E) &= \{u \in E : \mathcal{G}(\partial_1, \dots, \partial_n)u \in E\}, \\ \mathcal{G}_E u &= \mathcal{G}(\partial_1, \dots, \partial_n)u \quad \text{if } u \in D(\mathcal{G}_E). \end{aligned}$$

We prove that if (iii) holds, then for every $k = 1, 2, \dots$ the Cauchy problem

$$\frac{d}{dt}u(t) = \mathcal{G}(\partial_1, \dots, \partial_n)u(t) \quad \text{for } t \in [0, \infty[, \quad u(0) = u_0, \quad (\text{iv})$$

with given $u_0 \in D(\mathcal{G}_E^k)$ has a solution $u(\cdot) \in C^k([0, \infty[; E)$ which is unique in the class $C^1([0, \infty[; \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m))$. This solution is given by the formula

$$u(t) = S_t * u_0 \quad \text{for } t \in [0, \infty[. \quad (\text{v})$$

Examples of spaces E satisfying (iii) are given in Sec. 8.

Hyperbolic partial differential systems with constant coefficients

The matricial partial differential operator $\mathbb{1}_{m \times m} \otimes \partial_t - \mathcal{G}(\partial_1, \dots, \partial_n)$ on $\mathbb{R}^{1+n} = \{(t, x_1, \dots, x_n)\}$ is called *hyperbolic with respect to the coordinate t* if (ii) holds and the hyperplane $t = 0$ is non-characteristic for the operator. This last holds if and only if

the degree of the polynomial of $1 + n$ variables

$$P(\lambda, \zeta_1, \dots, \zeta_n) = \det(\lambda \mathbb{1}_{m \times m} - \mathcal{G}(\zeta_1, \dots, \zeta_n))$$

is equal to m . (vi)

Suppose that (ii) is satisfied and $(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ is the infinitely differentiable convolution semigroup whose generating distribution is $\mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta$. Then the question arises about properties of $(S_t)_{t \geq 0}$ corresponding to (vi). We prove that

- (a) if (vi) holds, then $(S_t)_{t \geq 0}$ extends to a one-parameter convolution group $(S_t)_{t \in \mathbb{R}}$ such that $\text{supp } S_t$ is bounded for every $t \in \mathbb{R}$, and
- (b) if (vi) does not hold, then $\text{supp } S_t$ is unbounded for every $t \in]0, \infty[$.

1 The setting and results

1.1 Notation

Throughout the present paper the symbols $\partial_1, \dots, \partial_n$ denote partial derivatives of the first order (not multiplied by any constant) of a function or distribution on \mathbb{R}^n . For partial derivatives of higher order we use the abbreviation $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$ where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ is a *multiindex* whose *length* is defined as $|\alpha| = \alpha_1 + \dots + \alpha_n$. $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ denote the space of *infinitely differentiable rapidly decreasing complex functions* on \mathbb{R}^n and the space of *slowly increasing distributions* on \mathbb{R}^n . The Fourier transformation

\mathcal{F} is defined by the formulas

$$\begin{aligned} (\mathcal{F}\varphi)(\xi_1, \dots, \xi_n) &= \hat{\varphi}(\xi_1, \dots, \xi_n) \\ &= \int \dots \int_{\mathbb{R}^n} e^{-i \sum_{k=1}^n x_k \xi_k} \varphi(x_1, \dots, x_n) dx_1 \dots dx_n \end{aligned} \quad (1.1)$$

whenever $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $(\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, and

$$\langle \mathcal{F}T, \varphi \rangle = \langle T, \mathcal{F}\varphi \rangle \quad (1.2)$$

whenever $T \in \mathcal{S}'(\mathbb{R}^n)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\mathcal{F}\varphi$ is determined by (1.1). The compatibility of (1.2) with (1.1) follows from the Parseval equality for a pair of elements of $\mathcal{S}(\mathbb{R}^n)$.

1.2 The function algebra $\mathcal{O}_M(\mathbb{R}^n)$ and the convolution algebra of distributions $\mathcal{O}'_C(\mathbb{R}^n)$

Let $\mathcal{O}_M(\mathbb{R}^n)$ be the space of *infinitely differentiable slowly increasing* complex functions on \mathbb{R}^n . Recall that $\phi \in \mathcal{O}_M(\mathbb{R}^n)$ if and only if for every $\alpha \in \mathbb{N}_0^n$ there is $m_\alpha \in \mathbb{N}_0$ such that

$$\sup_{\xi \in \mathbb{R}^n} (1 + |\xi|)^{-m_\alpha} |\partial^\alpha \phi(\xi)| < \infty.$$

Obviously $\mathcal{O}_M(\mathbb{R}^n)$ is a function algebra. Furthermore

$$\mathcal{O}_M(\mathbb{R}^n) = \{\phi \in C^\infty(\mathbb{R}^n) : \phi \cdot \varphi \in \mathcal{S}(\mathbb{R}^n) \text{ for every } \varphi \in \mathcal{S}(\mathbb{R}^n)\}. \quad (1.3)$$

For every $k \in \mathbb{N}_0$ denote by $\mathbf{B}_k(\mathbb{R}^n)$ the space of continuous complex functions f on \mathbb{R}^n such that $f(x) = O(|x|^{-k})$ as $|x| \rightarrow \infty$. A distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ is called *rapidly decreasing* if for every $k \in \mathbb{N}_0$ there is $m_k \in \mathbb{N}_0$ such that $T = \sum_{|\alpha| \leq m_k} \partial^\alpha f_{k,\alpha}$ where $f_{k,\alpha} \in \mathbf{B}_k(\mathbb{R}^n)$ for every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq m_k$. The space of rapidly decreasing distributions on \mathbb{R}^n , denoted by $\mathcal{O}'_C(\mathbb{R}^n)$, is a convolution algebra ^{*)}. One has

$$\mathcal{O}'_C(\mathbb{R}^n) = \{T \in \mathcal{S}'(\mathbb{R}^n) : T * \varphi \in \mathcal{S}(\mathbb{R}^n) \text{ for every } \varphi \in \mathcal{S}(\mathbb{R}^n)\}. \quad (1.4)$$

Since $\mathcal{O}_M(\mathbb{R}^n)$ and $\mathcal{O}'_C(\mathbb{R}^n)$ are subsets of $\mathcal{S}'(\mathbb{R}^n)$, $\mathcal{FO}_M(\mathbb{R}^n)$ and $\mathcal{FO}'_C(\mathbb{R}^n)$ make sense. Furthermore,

$$\mathcal{FO}'_C(\mathbb{R}^n) = \mathcal{O}_M(\mathbb{R}^n) \quad (1.5)$$

^{*)} See [S3, Sec. VII.5, pp. 246–248], [K-R, pp. 131–134].

and

$$\mathcal{F}(U * V) = (\mathcal{F}U) \cdot (\mathcal{F}V) \quad (1.6)$$

for every $U, V \in \mathcal{O}'_C(\mathbb{R}^n)$ ^{*}. The equality (1.6) means that the Fourier transformation is an (algebraic) isomorphism of the convolution algebra of distributions $\mathcal{O}'_C(\mathbb{R}^n)$ onto the function algebra $\mathcal{O}_M(\mathbb{R}^n)$.

By the closed graph theorem, it follows from (1.3) and (1.4) that the operators $\phi \cdot$ for $\phi \in \mathcal{O}_M(\mathbb{R}^n)$ and $T*$ for $T \in \mathcal{O}'_C(\mathbb{R}^n)$ belong to the space $L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$ of continuous linear operators from $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\mathbb{R}^n)$. Let $L_b(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$ denote the space $L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$ equipped with the compact-open topology. The sets of operators $\mathcal{O}_M(\mathbb{R}^n) \cdot$ and $\mathcal{O}'_C(\mathbb{R}^n) *$ are closed subspaces of $L_b(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$, and we treat them as equipped with the induced topology. The Fourier transformation is a continuous isomorphism of $\mathcal{O}'_C(\mathbb{R}^n)$ onto $\mathcal{O}_M(\mathbb{R}^n)$. Furthermore, the bilinear maps $\mathcal{O}_M(\mathbb{R}^n) \times \mathcal{O}_M(\mathbb{R}^n) \ni (\phi, \psi) \mapsto \phi \cdot \psi \in \mathcal{O}_M(\mathbb{R}^n)$ and $\mathcal{O}'_C(\mathbb{R}^n) \times \mathcal{O}'_C(\mathbb{R}^n) \ni (S, T) \mapsto S * T \in \mathcal{O}'_C(\mathbb{R}^n)$ are hypocontinuous. We shall prove the latter fact; the proof of the former is the same. Since $\mathcal{S}(\mathbb{R}^n)$ is a barrelled space, the boundedness of a subset of $L_b(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$ is equivalent to its equicontinuity. This implies that composition in $L_b(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$ is hypocontinuous. Since for $U, V \in \mathcal{O}'_C(\mathbb{R}^n)$ one has $(S*)|_{\mathcal{S}(\mathbb{R}^n)}, (V*)|_{\mathcal{S}(\mathbb{R}^n)} \in L(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$ and $((U * V)*)|_{\mathcal{S}(\mathbb{R}^n)} = (U*)|_{\mathcal{S}(\mathbb{R}^n)} \circ (V*)|_{\mathcal{S}(\mathbb{R}^n)}$, it follows that convolution in $\mathcal{O}'_C(\mathbb{R}^n)$ is hypocontinuous.

1.3 The function algebra $\mathcal{O}_M(\mathbb{R}^n; M_{m \times m})$ and the convolution algebra of distributions $\mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$

Let $m, n \in \mathbb{N}$, and let $M_{m \times m}$ be the set of $m \times m$ matrices with complex entries. Denote by $\mathcal{O}_M(\mathbb{R}^n; M_{m \times m})$ the space of functions of the form $\phi : \mathbb{R}^n \ni \xi \mapsto (\phi_{j,k}(\xi))_{j,k=1}^m \in M_{m \times m}$ such that $\phi_{j,k} \in \mathcal{O}_M(\mathbb{R}^n)$ for all j, k . This space carries the topology of $\mathcal{O}_M(\mathbb{R}^n)^{m^2}$ where each factor is equipped with the topology induced by $L_b(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$. Multiplication in $\mathcal{O}_M(\mathbb{R}^n; M_{m \times m})$ is defined by the rule

$$(\phi \cdot \psi)(\xi) = \left(\sum_{j=1}^m \phi_{i,j}(\xi) \psi_{j,k}(\xi) \right)_{i,k=1}^m.$$

$\mathcal{O}_M(\mathbb{R}^n; M_{m \times m})$ is a locally convex algebra with hypocontinuous multiplication.

^{*} See [S3, Sec. VII.8, Theorem XV, p. 268], [K-R, Theorem 8.23, p. 156].

Denote by $\mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ the space of $m \times m$ matrices $T = (T_{j,k})_{j,k=1}^m$ such that $T_{j,k} \in \mathcal{O}'_C(\mathbb{R}^n)$ for all j, k . The convolution in $\mathcal{O}_M(\mathbb{R}^n; M_{m \times m})$ is defined by the rule

$$S * T = \left(\sum_{j=1}^m S_{i,j} * T_{j,k} \right)_{i,k=1}^m.$$

The space $\mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ carries the topology of $\mathcal{O}'_C(\mathbb{R}^n)^{m^2}$ where each factor is equipped with the topology induced by $L_b(\mathcal{S}(\mathbb{R}^n); \mathcal{S}(\mathbb{R}^n))$. The l.c.v.s. $\mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ is a locally convex associative convolution algebra of $M_{m \times m}$ -valued distributions. Convolution in $\mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ is hypocontinuous.

The analogues of (1.5) and (1.6) are valid for $\mathcal{O}_M(\mathbb{R}^n; M_{m \times m})$ and $\mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$.

1.4 Infinitely differentiable one-parameter convolution semigroups in $\mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$

By a one-parameter *infinitely differentiable convolution semigroup* in $\mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$, briefly *i.d.c.s.*, we mean a mapping

$$[0, \infty[\ni t \mapsto S_t \in \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m}) \quad (1.7)$$

such that

$$S_{s+t} = S_s * S_t \text{ for every } s, t \in [0, \infty[, \quad (1.8)$$

$$S_0 = \mathbb{1}_{m \times m} \otimes \delta \text{ where } \mathbb{1}_{m \times m} \text{ is the unit } m \times m \text{ matrix and } \delta \text{ is the Dirac distribution on } \mathbb{R}^n, \quad (1.9)$$

$$\text{the mapping (1.7) is infinitely differentiable.} \quad (1.10)$$

In (1.10) it is understood that the derivatives at zero are right derivatives, and that the topology in $\mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ is that defined in Sec. 1.3.

The *generating distribution* of the i.d.c.s. $(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ is defined as

$$G := \left. \frac{d}{dt} \right|_{t=0} S_t \in \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m}).$$

It follows that

$$\frac{d}{dt} S_t = G * S_t = S_t * G \quad \text{for every } t \in [0, \infty[.$$

Furthermore, *any i.d.c.s. in $\mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ is uniquely determined by its generating distribution*. Indeed, suppose that $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ is the generating distribution of two i.d.c.s. $(S_t)_{t \geq 0}, (T_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$. Fix any $t \in$

$]0, \infty[$. Then $(S_\tau *)|_{\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m)}, (T_{t-\tau} *)|_{\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m)} \in L(\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m); \mathcal{S}(\mathbb{R}^n; \mathbb{C}^m))$ and

$$((S_\tau * T_{t-\tau}) *)|_{\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m)} = (S_\tau *)|_{\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m)} \circ (T_{t-\tau} *)|_{\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m)} \quad \text{for every } \tau \in [0, t].$$

Since $\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m)$ is a Montel (and hence barrelled) space, one infers from the Banach–Steinhaus theorem that the function $[0, t] \ni \tau \mapsto S_\tau * T_{t-\tau} \in \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ is continuously differentiable and

$$\frac{d}{d\tau}(S_\tau * T_{t-\tau}) = \left(\frac{d}{d\tau} S_\tau \right) * T_{t-\tau} + S_\tau * \left(\frac{d}{d\tau} T_{t-\tau} \right).$$

Consequently,

$$\frac{d}{d\tau}(S_\tau * T_{t-\tau}) = (S_\tau * G) * T_{t-\tau} - S_\tau * (G * T_{t-\tau}) = 0,$$

by associativity of the convolution in $\mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$, so that $S_\tau * T_{t-\tau}$ is independent of τ for $\tau \in [0, t]$, and $S_t = (S_\tau * T_{t-\tau})|_{\tau=t} = (S_\tau * T_{t-\tau})|_{\tau=0} = T_t$.

The Cauchy problem for a PDO with constant coefficients can be reduced by Fourier transformation with respect to the spatial coordinates to the Cauchy problem with a parameter for an ODO. In the framework of the spaces $\mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ and $\mathcal{O}_M(\mathbb{R}^n; M_{m \times m})$ this method consists in making use of the following

Lemma. *Suppose that $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ and let $A = \mathcal{F}G$, so that $A \in \mathcal{O}_M(\mathbb{R}^n; M_{m \times m})$. Then the following two conditions are equivalent:*

- (a) *G is the generating distribution of the i.d.c.s. $(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$,*
- (b) *$\exp(tA(\cdot)) \in \mathcal{O}_M(\mathbb{R}^n; M_{m \times m})$ for every $t \in [0, \infty[$ and the mapping $[0, \infty[\ni t \mapsto \exp(tA(\cdot)) \in \mathcal{O}_M(\mathbb{R}^n; M_{m \times m})$ is infinitely differentiable.*

Furthermore, if $A = \mathcal{F}G$ and (a), (b) are satisfied, then $\exp(tA(\cdot)) = \mathcal{F}S_t$ and

$$(S_t *)|_{\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)} = \mathcal{F}^{-1} \circ [(\exp tA(\cdot)) \cdot] \circ \mathcal{F}|_{\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)} \quad \text{for every } t \in [0, \infty[.$$

Basing on the above lemma we shall prove four theorems. For this purpose we shall use some intricate facts concerning \mathcal{O}'_C and \mathcal{O}_M , which for the most part are only mentioned in [S3], and are presented in detail in [K3]. For any $B \in M_{m \times m}$ denote by $\sigma(B)$ the spectrum of the matrix B .

Theorem 1. *A distribution $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ is the generating distribution of an i.d.c.s. $(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ if and only if*

$$\max\{\operatorname{Re} \lambda : \lambda \in \sigma((\mathcal{F}G)(\xi))\} = O(\log |\xi|) \quad \text{as } |\xi| \rightarrow \infty, \xi \in \mathbb{R}^n. \quad (1.11)$$

The quantity

$$s(G) := \sup\{\operatorname{Re} \lambda : \text{there is } \xi \in \mathbb{R}^n \text{ such that } \lambda \in \sigma((\mathcal{F}G)(\xi))\}, \quad (1.12)$$

finite or equal to $+\infty$, will be called the *spectral bound* of G . For any i.d.c.s. $(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ let

$$\omega((S_t)_{t \geq 0}) := \inf\{\omega \in \mathbb{R}^n : \text{the one-parameter semigroup of operators}$$

$$((e^{-\omega t} S_t *)|_{\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m)})_{t \geq 0} \subset L(\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m); \mathcal{S}(\mathbb{R}^n; \mathbb{C}^m))$$

$$\text{is equicontinuous}\} \quad (1.13)$$

where it is assumed that $\inf \emptyset = +\infty$. We call $\omega((S_t)_{t \geq 0})$ the *growth bound* of the i.d.c.s. $(S_t)_{t \geq 0}$ ^{*}).

Theorem 2. *For every i.d.c.s. $(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ its growth bound is equal to the spectral bound of its generating distribution.*

Let $\mathcal{G}(\partial_1, \dots, \partial_n)$ be an $m \times m$ matrix whose entries are PDOs on \mathbb{R}^n with constant complex coefficients. Let δ be the Dirac distribution on \mathbb{R}^n . Then $\mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta \in \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ and

$$[\mathcal{F}(\mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta)](\xi) = \mathcal{G}(i\xi_1, \dots, i\xi_n)$$

for every $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$. The quantity

$$s_0(\mathcal{G}) := \sup\{\operatorname{Re} \lambda : \lambda \in \sigma(\mathcal{G}(i\xi_1, \dots, i\xi_n)), (\xi_1, \dots, \xi_n) \in \mathbb{R}^n\}$$

is equal to the spectral bound of the distribution $\mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta$. It was conjectured by I. G. Petrovskii [P, footnote on p. 24] and proved by L. Gårding [G, Lemma on p. 11] that $s_0(\mathcal{G}) < \infty$ if and only if $G = \mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta$ satisfies (1.11) ^{**}). Therefore Theorems 1.1 and 1.2 imply

^{*}) In (1.13) the *growth bound with respect to $\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m)$* is defined. The growth bounds with respect to some other spaces invariant for the semigroup $(S_t *)_{t \geq 0}$ are also equal to the spectral bound of the generating distribution. See [B] and [K2, Theorem 1]. For one-parameter semigroups of operators in a Banach space the relations between the growth bound of the semigroup and the spectral bound of its generator are discussed in great detail in [E-N, Sec. IV.2].

^{**}) If $G = \mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta$, then (1.11) takes the form $\sup\{\operatorname{Re} \lambda : \lambda \in \sigma(\mathcal{G}(i\xi))\} = O(\log |\xi|)$ as $|\xi| \rightarrow \infty$, and in this form (1.11) occurs in [P, Sec. I.5]. However, usually the “Petrovskii condition” means the assumption that $s_0(\mathcal{G}) < \infty$.

Theorem 3. *Let $\mathcal{G}(\partial_1, \dots, \partial_n)$ be an $m \times m$ matrix whose entries are PDOs on \mathbb{R}^n with constant complex coefficients. Then the following two conditions are equivalent:*

$$s_0(\mathcal{G}) < \infty, \quad (1.14)$$

$$\begin{aligned} &\mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta \text{ is the generating distribution of an i.d.c.s.} \\ &(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m}). \end{aligned} \quad (1.15)$$

Furthermore, if these equivalent conditions are fulfilled, then there is exactly one i.d.c.s. $(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ satisfying (1.15), and the growth bound of this i.d.c.s. is equal to $s_0(\mathcal{G})$.

Example 1. Let $m = 1$, $\mathcal{G}(\partial_1, \dots, \partial_n) = i(\partial_1^2 + \dots + \partial_n^2)$. Then $s_0(\mathcal{G}) = s_0(-\mathcal{G}) = 0$, so that $\mathcal{G}(\partial_1, \dots, \partial_n)\delta$ and $-\mathcal{G}(\partial_1, \dots, \partial_n)\delta$ are generating distributions of i.d.c.s. imbedded in $\mathcal{O}'_C(\mathbb{R}^n)$. Consequently, $i(\partial_1^2 + \dots + \partial_n^2)\delta$ is the generating distribution of an *infinitely differentiable one-parameter convolution group* $(S_t)_{t \in \mathbb{R}^n} \subset \mathcal{O}'_C(\mathbb{R}^n)$. This group of distributions satisfies the Schrödinger partial differential equation

$$\partial_t S_t = i(\partial_1^2 + \dots + \partial_n^2) S_t,$$

one has $S_0 = \delta$, and for every $t \in \mathbb{R}^n \setminus \{0\}$ the distribution S_t is equal to the bounded function belonging to $\mathcal{O}_M(\mathbb{R}^n)$ such that

$$S_t(x) = (4\pi it)^{-n/2} \exp\left(\frac{i|x|^2}{4t}\right) \quad \text{whenever } x \in \mathbb{R}^n.$$

The factor $(4\pi it)^{-n/2}$ is defined as $\left(\frac{1}{\sqrt{4\pi it}}\right)^n$ where $\arg \sqrt{4\pi it} = (\pi/4) \operatorname{sgn} t$. See [Go, p. 54], [R, p. 107], [S1, p. 48]. The direct proof that $S_t \subset \mathcal{O}'_C(\mathbb{R}^n)$ is on p. 245 of [S2]. Another proof is by Fourier transformation: one has $(\mathcal{F}S_t)(\xi) = e^{-it|\xi|^2}$, so that $\mathcal{F}S_t \in \mathcal{O}_M(\mathbb{R}^n)$, and hence $S_t \subset \mathcal{O}'_C(\mathbb{R}^n)$.

Example 2. Following J. Rauch [R, Sec. 3.10] we look for solutions of class $C^\infty([0, \infty[; \mathcal{S}(\mathbb{R}^n))$ of the Cauchy problem

$$\begin{aligned} \sum_{k=0}^m Q_k(\partial_1, \dots, \partial_n) \partial_t^k u(t, x_1, \dots, x_n) &= 0 \\ &\text{for } (t, x_1, \dots, x_n) \in [0, \infty[\times \mathbb{R}^n, \\ \partial_t^k u(0, x_1, \dots, x_n) &= u_k(x_1, \dots, x_n) \\ &\text{for } k = 0, \dots, m-1 \text{ and } (x_1, \dots, x_n) \in \mathbb{R}^n, \end{aligned} \quad (1.16)$$

where $Q_k(\partial_1, \dots, \partial_n)$, $k = 0, \dots, m$, are linear partial differential operators with constant coefficients, and $u_k \in \mathcal{S}(\mathbb{R}^n)$, $k = 0, \dots, m-1$, are given. As in [R, Sec. 3.10], we assume that the polynomial

$$P(\lambda, \zeta_1, \dots, \zeta_n) = Q_m(\zeta_1, \dots, \zeta_n)\lambda^m + \dots + Q_1(\zeta_1, \dots, \zeta_n)\lambda + Q_0(\zeta_1, \dots, \zeta_n)$$

has two properties:

$$\sup\{\operatorname{Re} \lambda : \lambda \in \mathbb{C}, \text{ there is } \xi \in \mathbb{R}^n \text{ such that } P(\lambda, i\xi) = 0\} = s_0 < \infty, \quad (1.17)$$

$$Q_m(i\xi) \neq 0 \text{ whenever } \xi \in \mathbb{R}^n. \quad (1.18)$$

For every $\xi \in \mathbb{R}^n$ denote by $\hat{G}(\xi)$ the matrix

$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & & & \\ & & 1 & & \\ & & 0 & & \\ -\frac{Q_0(i\xi)}{Q_m(i\xi)} & -\frac{Q_1(i\xi)}{Q_m(i\xi)} & \dots & -\frac{Q_{m-2}(i\xi)}{Q_m(i\xi)} & -\frac{Q_{m-1}(i\xi)}{Q_m(i\xi)} \end{bmatrix}.$$

By [H, Example A.2.7] there is $m_0 \in \mathbb{N}$ such that

$$\sup\{(1 + |\xi|)^{-m_0} |Q_m(i\xi)^{-1}| : \xi \in \mathbb{R}^n\} < \infty.$$

Since $\partial^\alpha(Q_m(i\xi)^{-1}) = Q_m(i\xi)^{-1-|\alpha|}R(\xi)$ for every $\alpha \in \mathbb{N}_0^n$ where $R(\xi)$ is a polynomial, it follows that $Q_m(i \cdot)^{-1} \in \mathcal{O}_M(\mathbb{R}^n)$. Consequently,

$$\hat{G} \in \mathcal{O}_M(\mathbb{R}^n; M_{m \times m}). \quad (1.19)$$

Furthermore,

$$\det(\lambda \mathbb{1}_{m \times m} - \hat{G}(\xi)) = \lambda^m + \frac{Q_{m-1}(i\xi)}{Q_m(i\xi)}\lambda^{m-1} + \dots + \frac{Q_1(i\xi)}{Q_m(i\xi)}\lambda + \frac{Q_0(i\xi)}{Q_m(i\xi)},$$

whence

$$\sigma(\hat{G}(\xi)) = \{\lambda \in \mathbb{C} : P(\lambda, i\xi) = 0\},$$

and so (1.17) implies that

$$\sup\{\operatorname{Re} \lambda : \lambda \in \sigma(\hat{G}(\xi)), \xi \in \mathbb{R}^n\} = s_0. \quad (1.20)$$

From (1.19) it follows that there is a unique distribution $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ such that $\mathcal{F}G = \hat{G}$. By Theorems 1 and 2, (1.20) implies that G is the

generating distribution of an i.d.c.s. $(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ such that $\omega((S_t)_{t \geq 0}) = s_0$ and

$$\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & Q_m & \end{bmatrix} \partial_t S_t = \begin{bmatrix} 0 & 1 & & & \\ & 0 & & & \\ & & & 1 & \\ & & & 0 & 1 \\ -Q_0 & -Q_1 & \cdots & -Q_{m-2} & -Q_{m-1} \end{bmatrix} S_t$$

for every $t \in [0, \infty[$ where $Q_k = Q_k(\partial_1, \dots, \partial_n)$ for $k = 0, \dots, m$. By arguments similar to that presented in Sec. 8, the above implies that, under the assumptions (1.17) and (1.18), for every $u_0, \dots, u_{m-1} \in \mathcal{S}(\mathbb{R}^n)$ and $u \in C^\infty([0, \infty[; \mathcal{S}(\mathbb{R}^n))$ the following two conditions are equivalent:

(a) u is a solution of the Cauchy problem (1.16),

$$(b) \begin{bmatrix} u(t, \cdot) \\ \partial_t u(t, \cdot) \\ \vdots \\ \partial_t^{m-1} u(t, \cdot) \end{bmatrix} = S_t * \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{m-1} \end{bmatrix} \text{ for } t \in [0, \infty[.$$

If only the condition (1.17) is satisfied and (1.18) may fail, then the i.d.c.s.'s in $\mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ seem not to be useful, but $\mathcal{O}'_C(\mathbb{R}^{1+n})$ can be used to express the properties of the fundamental solution for the operator $Q_m(\partial_1, \dots, \partial_n) \partial_t^m + \cdots + Q_1(\partial_1, \dots, \partial_n) \partial_t + Q_0(\partial_1, \dots, \partial_n)$ with support contained in $H_+ = \{(t, x_1, \dots, x_n) \in \mathbb{R}^{1+n} : t \geq 0\}$. See the article of the present author in arXiv:1105.0877.

Comments. I. G. Petrovskiĭ [P] was the first to notice the significance of smooth slowly increasing functions in the theory of evolutionary PDEs. The theory of distributions did not yet exist in 1938 when [P] was published, and only in 1950 did L. Schwartz explain in [S1] how the results of Petrovskiĭ may be elucidated by placing them in the framework of \mathcal{O}'_C . However in [S1] the spectral properties of $[\mathcal{F}(\mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta)](\xi) = \mathcal{G}(i\xi, \dots, i\xi_n)$ were not discussed.

If $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ and $\mathcal{N}_G = \{(\lambda, \xi) \in \mathbb{C} \times \mathbb{R}^n : \det(\lambda \mathbb{1}_{m \times m} - (\mathcal{F}G)(\xi)) = 0\}$, then (1.11) may be expressed in an equivalent form: there is $C \in]0, \infty[$ such that

$$\text{if } (\lambda, \xi) \in \mathcal{N}_G, \text{ then } \operatorname{Re} \lambda \leq C(1 + \log(1 + |\xi|)). \quad (1.11)'$$

As mentioned earlier, just this logarithmic condition was used in [P]. In connection with convolution equations similar logarithmic estimates (in \mathbb{C}^{1+n} instead of $\mathbb{C} \times \mathbb{R}^n$) were used by L. Ehrenpreis in [E1] and in [E2, Sec. VIII.3].

Logarithmic estimates related to convolution equations also occur in elaborate theorems of L. Hörmander [H, Secs. 16.6 and 16.7]. The role of conditions (1.14) and (1.11) in the theory of evolutionary PDOs with constant coefficients is discussed in [R, Sec. 3.10].

From the above-mentioned Petrovskii conjecture proved by Gårding, and from Theorem 3, it follows that whenever the generating distribution $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ of an i.d.c.s. $(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ has the form $G = \mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta$, then

$$\sup\{\operatorname{Re} \lambda : \lambda \in \sigma(\mathcal{G}(i\xi_1, \dots, i\xi_n)), (\xi_1, \dots, \xi_n) \in \mathbb{R}^n\} = s_0 < \infty,$$

and whenever $\varepsilon > 0$, then the semigroup of operators

$$((e^{-(s_0+\varepsilon)t} S_t *)|_{\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m)})_{t \geq 0} \subset L(\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m); \mathcal{S}(\mathbb{R}^n; \mathbb{C}^m))$$

is equicontinuous. As noticed by L. Schwartz [S2], the theory of equicontinuous one-parameter semigroups of operators in an l.c.v.s. imitates the theory of one-parameter semigroups of operators in a Banach space. A detailed presentation of the theory of equicontinuous one-parameter semigroups of operators in a sequentially complete l.c.v.s. is contained in Chapter IX of the monograph of K. Yosida [Y].

1.5 Relation to hyperbolic systems of PDOs

Let $\mathcal{E}'(\mathbb{R}^n)$ be the space of distributions on \mathbb{R}^n with compact support, equipped with the topology of uniform convergence on bounded subsets of $C^\infty(\mathbb{R}^n)$. L. Ehrenpreis [E2, Sec. V.5] proved that $\mathcal{E}'(\mathbb{R}^n) = \{T \in \mathcal{D}'(\mathbb{R}^n) : T * \in L(\mathcal{D}(\mathbb{R}^n); \mathcal{D}(\mathbb{R}^n))\}$ and the topology induced in $\mathcal{E}'(\mathbb{R}^n)$ by $L_b(\mathcal{D}(\mathbb{R}^n); \mathcal{D}(\mathbb{R}^n))$ via the mapping $T \mapsto T *$ coincides with the original topology of $\mathcal{E}'(\mathbb{R}^n)$. This topology is stronger than the one induced on $\mathcal{E}'(\mathbb{R}^n)$ by $\mathcal{O}'_C(\mathbb{R}^n)$. See [S3, Sec. III.7], [E2, Sec. V.5, Lemma 5.17]. Let $\mathcal{E}'(\mathbb{R}^n; M_{m \times m})$ be the space of $M_{m \times m}$ -valued distributions on \mathbb{R}^n with compact support, i.e. the space of $m \times m$ matrices whose entries belong to $\mathcal{E}'(\mathbb{R}^n)$. With the topology of $\mathcal{E}'(\mathbb{R}^n)^{m^2}$ and convolution defined as in $\mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$, the space $\mathcal{E}'(\mathbb{R}^n; M_{m \times m})$ is a convolution algebra with continuous convolution. See [S3, Sec. VII.3, Theorem IV].

As in Theorem 1.3, let $\mathcal{G}(\partial_1, \dots, \partial_n)$ be an $m \times m$ matrix whose entries are PDOs on \mathbb{R}^n with constant complex coefficients. Put

$$P(\lambda, \zeta_1, \dots, \zeta_n) = \det(\lambda \mathbb{1}_{m \times m} - \mathcal{G}(\zeta_1, \dots, \zeta_n)) \quad (1.21)$$

where $(\lambda, \zeta_1, \dots, \zeta_n) \in \mathbb{C}^{1+n}$.

Theorem 4. Assume that $\mathcal{G}(\partial_1, \dots, \partial_n)$ satisfies condition (1.14), and let $(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ be the i.d.c.s. whose generating distribution is $\mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta$. Then the following three conditions are equivalent:

$$\text{there is } t_0 \in]0, \infty[\text{ such that } S_{t_0} \in \mathcal{E}'(\mathbb{R}^n; M_{m \times m}), \quad (1.22)$$

$$\text{the polynomial } P(\lambda, \zeta_1, \dots, \zeta_n) \text{ defined by (1.21) has degree } m, \quad (1.23)$$

$$(S_t)_{t \geq 0} \text{ is an i.d.c.s. in the topological convolution algebra } \mathcal{E}'(\mathbb{R}^n; M_{m \times m}), \text{ and may be uniquely extended to a one-parameter infinitely differentiable subgroup of } \mathcal{E}'(\mathbb{R}^n; M_{m \times m}). \quad (1.24)$$

Let

$$N = \{(\lambda, \zeta_1, \dots, \zeta_n) \in \mathbb{C}^{1+n} : P(\lambda, \zeta_1, \dots, \zeta_n) = 0\}. \quad (1.25)$$

The matricial PDO

$$\mathbb{1}_{m \times m} \otimes \partial_t - \mathcal{G}(\partial_1, \dots, \partial_n) \quad (1.26)$$

on $\mathbb{R}^{1+n} = \{(t, x_1, \dots, x_n) : t \in \mathbb{R}, (x_1, \dots, x_n) \in \mathbb{R}^n\}$ is said to be *hyperbolic in the sense of Ehrenpreis with respect to the coordinate t* if there is $C \in]0, \infty[$ such that

$$\text{if } (\lambda, \zeta_1, \dots, \zeta_n) \in N, \text{ then } |\operatorname{Re} \lambda| \leq C(1 + |\operatorname{Re} \zeta_1| + \dots + |\operatorname{Re} \zeta_n|). \quad (1.27)$$

Condition (1.27) is stronger than (1.14) which is equivalent to the existence of $C \in]0, \infty[$ such that

$$(1.14)' \quad \text{if } (\lambda, \zeta_1, \dots, \zeta_n) \in N \text{ and } \operatorname{Re} \zeta_1 = \dots = \operatorname{Re} \zeta_n = 0, \text{ then } \operatorname{Re} \lambda \leq C.$$

The matricial PDO (1.26) is said to be *hyperbolic in the sense of Gårding with respect to the coordinate t* if the polynomial (1.21) satisfies (1.14)' and (1.23). In the proof of Theorem 4 it will be shown that for the matricial PDO (1.26) these two notions of hyperbolicity with respect to t are equivalent. Therefore Theorem 4 may be reformulated as follows: *if $\mathcal{G}(\partial_1, \dots, \partial_n)$ satisfies the Petrovskii condition (1.14), then for the semigroup $(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ with generating distribution $\mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta$ the properties (1.22) and (1.24) are equivalent, and they both hold if and only if the matricial PDO (1.26) is hyperbolic with respect to the variable t .*

Suppose that (1.26) is hyperbolic with respect to t . Let P_m be the principal homogeneous part of the polynomial (1.21), and let Γ be the connected component of the set $\{(\sigma, \xi_1, \dots, \xi_n) \in \mathbb{R}^{1+n} : P_m(\sigma, \xi_1, \dots, \xi_n) \neq 0\}$ which contains $(1, 0, \dots, 0)$. By [H, Lemma 8.7.3], Γ is a convex cone. Let Γ^0 be the closed cone dual to Γ . Using [H, Theorem 12.5.1] it may be proved that

$$\Gamma^0 = \{(t, x_1, \dots, x_n) \in \mathbb{R}^{1+n} : t \geq 0, (x_1, \dots, x_n) \in \operatorname{conv} \operatorname{supp} S_t\} \quad (1.28)$$

where $(S_t)_{t \geq 0}$ is the i.d.c.s. occurring in Theorem 4. By (1.28), the distribution $N \in \mathcal{D}'(\mathbb{R}^n; M_{m \times m})$ such that $\langle N, \varphi \rangle = \int_0^\infty \langle S_t, \varphi(t, \cdot) \rangle dt$ for every $\varphi \in \mathcal{D}(\mathbb{R}^{1+n})$ is a fundamental solution of (1.26) with support contained in Γ^0 . Theorem 4 resembles Theorems V and VI of [S1, Sec. 13], and Theorems 12.5.1 and 12.5.2 of [H].

2 A link between properties of $M_{m \times m}$ -valued functions $\xi \mapsto A(\xi)$ and $(t, \xi) \mapsto \exp(tA(\xi))$

Theorem 2.1 (The Shilov inequality). *Let $A \in M_{m \times m}$. Then for every $t \in [0, \infty[$ one has*

$$\|\exp(tA)\|_{M_{m \times m}} \leq \rho(\exp(tA)) \left(1 + \sum_{k=1}^{m-1} \frac{(2t)^k}{k!} \|A\|_{M_{m \times m}}^k \right) \quad (2.1)$$

and

$$\rho(\exp(tA)) = e^{t \max \operatorname{Re} \sigma(A)} \quad (2.2)$$

where ρ stands for the spectral radius, and $\sigma(A)$ denotes the spectrum of A .

The equality (2.2) follows from the spectral mapping theorem. The Shilov inequality (2.1) is an elaborate result of the theory of functions of matrices. See [Sh], [Ge, Sec. I.4], [G-S, Sec. II.6], [F, Sec. 7.2]. We say that $\Phi \subset C^\infty(\mathbb{R}^n; M_{m \times m})$ is a *set of uniformly slowly increasing functions* if for every $\alpha \in \mathbb{N}_0^n$ there is $k_\alpha \in \mathbb{N}_0$ such that $\sup\{(1 + |\xi|)^{-k_\alpha} \|(\partial/\partial\xi)^\alpha \phi(\xi)\|_{M_{m \times m}} : \phi \in \Phi, \xi \in \mathbb{R}^n\} < \infty$.

Proposition 2.2. *For any $A(\cdot) \in \mathcal{O}_M(\mathbb{R}^n; M_{m \times m})$ the following three conditions are equivalent:*

$$\max \operatorname{Re} \sigma(A(\xi)) = O(\log |\xi|) \text{ as } |\xi| \rightarrow \infty, \quad (2.3)$$

for every $T \in]0, \infty[$ there are $C \in]0, \infty[$ and $k \in \mathbb{N}$ such that

$$\|\exp(tA(\xi))\|_{M_{m \times m}} \leq C(1 + |\xi|)^k \quad (2.4)$$

whenever $t \in [0, T]$ and $\xi \in \mathbb{R}^n$,

whenever $T \in]0, \infty[$, then $\{\exp(tA(\cdot)) : t \in [0, T]\}$ is a set of uniformly slowly increasing infinitely differentiable $M_{m \times m}$ -valued functions on \mathbb{R}^n . (2.5)

Proposition 2.3. *For every $A(\cdot) \in \mathcal{O}_M(\mathbb{R}^n; M_{m \times m})$ and $s_0 \in \mathbb{R}$ the following five conditions are equivalent:*

$$\sup\{\operatorname{Re} \lambda : \lambda \in \sigma(A(\xi)), \xi \in \mathbb{R}^n\} \leq s_0; \quad (2.6)$$

there is $k \in \mathbb{N}_0$ such that for every $\varepsilon > 0$,

$$\sup\{e^{-(s_0+\varepsilon)t}(1+|\xi|)^{-k}\|\exp(tA(\xi))\|_{M_{m \times m}} : t \in [0, \infty[, \xi \in \mathbb{R}^n\} < \infty; \quad (2.7)$$

for every $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that

$$\sup\{e^{-(s_0+\varepsilon)t}(1+|\xi|)^{-k}\|\exp(tA(\xi))\|_{M_{m \times m}} : t \in [0, \infty[, \xi \in \mathbb{R}^n\} < \infty; \quad (2.7)^*$$

for every $\alpha \in \mathbb{N}_0^n$ there is $k_\alpha \in \mathbb{N}_0$ such that for every $\varepsilon > 0$,

$$\sup\{e^{-(s_0+\varepsilon)t}(1+|\xi|)^{-k_\alpha}\|(\partial/\partial\xi)^\alpha \exp(tA(\xi))\|_{M_{m \times m}} : t \in [0, \infty[, \xi \in \mathbb{R}^n\} < \infty; \quad (2.8)$$

whenever $\varepsilon \in]0, \infty[$, then $\{e^{-(s_0+\varepsilon)t} \exp(tA(\cdot)) : t \in [0, \infty[\}$ is a set of uniformly slowly increasing infinitely differentiable $M_{m \times m}$ -valued functions on \mathbb{R}^n . (2.8)*

Our proofs of Propositions 2.2 and 2.3 are based on the Shilov inequality. In [P, Sec. I.5], in the proof of the prototype of Proposition 2.2, instead of the Shilov inequality, I. G. Petrovskii used [P, Sec. I.5, Lemma 5]. We shall prove Propositions 2.2 and 2.3 according to the schemes (2.3) \Rightarrow (2.4) \Rightarrow (2.5) \Rightarrow (2.4) \Rightarrow (2.3) and (2.6) \Rightarrow (2.7) \Rightarrow (2.8) \Rightarrow (2.8)* \Rightarrow (2.7)* \Rightarrow (2.6) where the implications (2.5) \Rightarrow (2.4) and (2.8) \Rightarrow (2.8)* \Rightarrow (2.7)* are trivial.

Proof of (2.3) \Leftrightarrow (2.4). If $A(\cdot) \in \mathcal{O}_M(\mathbb{R}^n; M_{m \times m})$ and (2.3) holds, then, by (2.1) and (2.2), for any fixed $T \in]0, \infty[$ there are $C, D \in]0, \infty[$ and $l \in \mathbb{N}_0$ such that for every $(t, \xi) \in [0, T] \times \mathbb{R}^n$ one has

$$\begin{aligned} \|\exp(tA(\xi))\|_{M_{m \times m}} &\leq e^{t \max \operatorname{Re} \sigma(A(\xi))} \left(1 + \sum_{k=1}^{m-1} \frac{(2t)^k}{k!} \|A(\xi)\|_{M_{m \times m}}^k \right) \\ &\leq e^{TC(1+\log(1+|\xi|))} (1 + 2T\|A(\xi)\|_{M_{m \times m}})^{m-1} \\ &\leq D(1+|\xi|)^{TC+l(m-1)}, \end{aligned}$$

so that (2.4) is satisfied. Conversely, if (2.4) holds, then there are $C \in]0, \infty[$ and $k \in \mathbb{N}_0$ such that $\|\exp A(\xi)\|_{M_{m \times m}} \leq C(1 + |\xi|)^k$ for every $\xi \in \mathbb{R}^n$, whence, by (2.2),

$$\begin{aligned} \max \operatorname{Re} \sigma(A(\xi)) &= \log \rho(\exp A(\xi)) \\ &\leq \log \|\exp A(\xi)\|_{M_{m \times m}} \leq \log C + k \log(1 + |\xi|), \end{aligned}$$

so that (2.3) holds.

Proof of (2.6) \Rightarrow (2.7). If (2.6) holds, then, by (2.1) and (2.2), for every $t \in]0, \infty[$ and $\xi \in \mathbb{R}^n$ one has

$$\begin{aligned} \|\exp(tA(\xi))\|_{M_{m \times m}} &\leq e^{s_0 t} \left(1 + \sum_{k=1}^{m-1} \frac{(2t)^k}{k!} \|A(\xi)\|_{M_{m \times m}}^k \right) \\ &\leq e^{s_0 t} (1 + 2t)^{m-1} (1 + \|A(\xi)\|_{M_{m \times m}})^{m-1}. \end{aligned}$$

Furthermore, since $A(\cdot) \in \mathcal{O}_M(\mathbb{R}^n; M_{m \times m})$, there are $C \in]0, \infty[$ and $l \in \mathbb{N}_0$ such that $\|A(\xi)\|_{M_{m \times m}} \leq C(1 + |\xi|)^l$ for every $\xi \in \mathbb{R}^n$. The above inequalities imply (2.7).

Proof of (2.7) \Rightarrow (2.6).* By (2.2),

$$\max \operatorname{Re} \sigma(A(\xi)) = \frac{1}{t} \log \rho(\exp(tA(\xi))) \leq \frac{1}{t} \log \|\exp(tA(\xi))\|_{M_{m \times m}}$$

for every $t \in]0, \infty[$ and $\xi \in \mathbb{R}^n$. So, if (2.7)* holds, then for every $\varepsilon > 0$ there are $C \in]0, \infty[$ and $k \in \mathbb{N}$ such that

$$\max \operatorname{Re} \sigma(A(\xi)) \leq s_0 + \varepsilon + \frac{1}{t} \log(C(1 + |\xi|)^k)$$

for every $t \in]0, \infty[$ and $\xi \in \mathbb{R}^n$, whence (2.6) follows.

Proof of (2.4) \Rightarrow (2.5) and (2.7) \Rightarrow (2.8). The proofs of these implications are similar, and both base on the argument of I. G. Petrovskiĭ from the proof of [P, Sec. I.2, Lemma 2]. We shall limit ourselves to (2.7) \Rightarrow (2.8).

For every $\alpha \in \mathbb{N}_0^n$ let

$$U_{\alpha, t}(\xi) = (\partial/\partial \xi)^\alpha \exp(tA(\xi)).$$

Consider the condition

there is $k_\alpha \in \mathbb{N}_0$ such that for every $\varepsilon > 0$ there is $C_{\alpha, \varepsilon}$ in $]0, \infty[$ such that whenever $(t, \xi) \in [0, \infty[\times \mathbb{R}^n$, then

$$\|U_{\alpha, t}(\xi)\|_{M_{m \times m}} \leq C_{\alpha, \varepsilon} e^{(s_0 + \varepsilon)t} (1 + |\xi|)^{k_\alpha}. \quad (2.9)_\alpha$$

Then (2.7) means that $(2.9)_0$ holds, and (2.8) means that $(2.9)_\alpha$ holds for every $\alpha \in \mathbb{N}_0^n$. So, still assuming that $A(\cdot) \in \mathcal{O}_M(\mathbb{R}^n; M_{m \times m})$, we have to prove that $(2.9)_0$ implies $(2.9)_\alpha$ for every $\alpha \in \mathbb{N}_0^n$. We proceed by induction on the length of α . By (2.7), $(2.9)_0$ is satisfied. Suppose that $(2.9)_\beta$ is satisfied whenever $|\beta| \leq l$, and take $\alpha \in \mathbb{N}_0^n$ such that $|\alpha| = l + 1$. To prove $(2.9)_\alpha$, put

$$V_{\alpha,t}(\xi) = \sum_{\beta \leq \alpha, |\beta| \leq l} \binom{\alpha}{\beta} \left(\left(\frac{\partial}{\partial \xi} \right)^{\alpha - \beta} A(\xi) \right) U_{\beta,t}(\xi).$$

Since $A(\cdot) \in \mathcal{O}_M(\mathbb{R}^n; M_{m \times m})$ and $(2.9)_\beta$ holds whenever $|\beta| \leq l$, it follows that

there is $h_\alpha \in \mathbb{N}_0$ such that for every $\varepsilon > 0$ there is $D_{\alpha,\varepsilon} \in]0, \infty[$ such that whenever $(t, \xi) \in [0, \infty[\times \mathbb{R}^n$, then

$$\|V_{\alpha,t}(\xi)\|_{M_{m \times m}} \leq D_{\alpha,\varepsilon} e^{(s_0 + \varepsilon)t} (1 + |\xi|)^{h_\alpha}. \quad (2.10)_\alpha$$

One has

$$\begin{aligned} \frac{\partial}{\partial t} U_{\alpha,t}(\xi) &= \frac{\partial}{\partial t} \left(\frac{\partial}{\partial \xi} \right)^\alpha \exp(tA(\xi)) = \left(\frac{\partial}{\partial \xi} \right)^\alpha [A(\xi) \exp(tA(\xi))] \\ &= A(\xi) U_{\alpha,t}(\xi) + V_{\alpha,t}(\xi) \end{aligned}$$

and $U_{\alpha,0}(\xi) = 0$ because $|\alpha| = l + 1 \geq 1$. Hence

$$U_{\alpha,t}(\xi) = \int_0^t [\exp((t - \tau)A(\xi))] V_{\alpha,t}(\xi) d\tau. \quad (2.11)$$

From $(2.9)_0$, $(2.10)_\alpha$ and (2.11) it follows that

$$\begin{aligned} \|U_{\alpha,t}(\xi)\|_{M_{m \times m}} &\leq \int_0^t C_{0,\varepsilon/2} e^{(s_0 + \varepsilon/2)(t - \tau)} (1 + |\xi|)^{k_0} D_{\alpha,\varepsilon/2} e^{(s_0 + \varepsilon/2)\tau} (1 + |\xi|)^{h_\alpha} d\tau \\ &= C_{0,\varepsilon/2} D_{\alpha,\varepsilon/2} t e^{(s_0 + \varepsilon/2)t} (1 + |\xi|)^{k_0 + h_\alpha} \leq \tilde{C}_{\alpha,\varepsilon} e^{(s_0 + \varepsilon)t} (1 + |\xi|)^{k_\alpha} \end{aligned}$$

for $k_\alpha = k_0 + h_\alpha$ and $\tilde{C}_{\alpha,\varepsilon} = C_{0,\varepsilon/2} D_{\alpha,\varepsilon/2} \max_{t \in [0, \infty[} t e^{-(\varepsilon/2)t}$.

3 Proof of Theorem 1

Necessity of (1.11). Suppose that $(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ is an i.d.c.s. with generating distribution $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$. Let $A = \mathcal{F}G$. Then $A, \mathcal{F}S_t \in \mathcal{O}_M(\mathbb{R}^n; M_{m \times m})$ and $(\mathcal{F}S_t)(\xi) = \exp(tA(\xi))$ for every $t \in [0, \infty[$ and $\xi \in \mathbb{R}^n$. Since the mapping $[0, \infty[\ni t \mapsto [\exp(tA(\cdot))] \cdot = (\mathcal{F}S_t) \cdot \in$

$L_b(\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m); \mathcal{S}(\mathbb{R}^n; \mathbb{C}^m))$ is continuous, the Banach–Steinhaus theorem implies that whenever $T \in]0, \infty[$, then the set of multiplication operators $\{[\exp(tA(\cdot))] \cdot : t \in [0, T]\}$ is an equicontinuous subset of $L(\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m); \mathcal{S}(\mathbb{R}^n; \mathbb{C}^m))$. By [K3, Theorem 3.1], this is equivalent to (2.5). By Proposition 2.2, (2.5) is equivalent to (2.3). Since $A = \mathcal{F}G$, (2.3) is nothing but (1.11).

Sufficiency of (1.11). Suppose that $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ satisfies (1.11). Let $A = \mathcal{F}G$. Then $A \in \mathcal{O}_M(\mathbb{R}^n; M_{m \times m})$, and A satisfies (2.3). Hence, by Proposition 2.2 and [K3, Theorem 3.1], whenever $T \in]0, \infty[$, then $\{[\exp(tA(\cdot))] \cdot : t \in [0, T]\}$ is an equicontinuous subset of $L(\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m); \mathcal{S}(\mathbb{R}^n; \mathbb{C}^m))$. By the theorem on differentiating a solution of an ODE with respect to a parameter [Ha, Sec. V.4, Corollary 4.1], the mapping $\mathbb{R}^{1+n} \ni (t, \xi) \mapsto \exp(tA(\xi)) \in M_{m \times m}$ is infinitely differentiable, and hence, by [K3, Theorem 3.2], so is $[0, \infty[\ni t \mapsto [\exp(tA(\cdot))] \cdot \in L_b(\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m); \mathcal{S}(\mathbb{R}^n; \mathbb{C}^m))$, and its right derivative at zero (computed in the topology of $L_b(\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m); \mathcal{S}(\mathbb{R}^n; \mathbb{C}^m))$) is $A \cdot \in L(\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m); \mathcal{S}(\mathbb{R}^n; \mathbb{C}^m))$. It follows that $G * = (\mathcal{F}^{-1}A) * = \mathcal{F}^{-1} \circ (A \cdot) \circ \mathcal{F} \in L(\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m); \mathcal{S}(\mathbb{R}^n; \mathbb{C}^m))$ is the infinitesimal generator of the infinitely differentiable operator semigroup $([\mathcal{F}^{-1} \exp(tA(\cdot))] *)_{t \geq 0} = (\mathcal{F}^{-1} \circ [\exp(tA(\cdot))] \cdot \circ \mathcal{F})_{t \geq 0} \subset L_b(\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m); \mathcal{S}(\mathbb{R}^n; \mathbb{C}^m))$. Consequently, $G = \mathcal{F}^{-1}A$ is the generating distribution of the i.d.c.s. $(\mathcal{F}^{-1} \exp(tA(\cdot)))_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$.

4 Proof of Theorem 2

Let $(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ be an i.d.c.s. with generating distribution $G \in \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$. Put $A = \mathcal{F}G$. Then $A, \mathcal{F}S_t \in \mathcal{O}_M(\mathbb{R}^n; M_{m \times m})$, condition (b) from the Lemma from Sec. 1.4 is satisfied, and $(\exp(tA(\cdot))) \cdot = \mathcal{F} \circ (S_t *) \circ \mathcal{F}^{-1}$ for every $t \in [0, \infty[$. Since $\mathcal{F}, \mathcal{F}^{-1} \in L(\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m); \mathcal{S}(\mathbb{R}^n; \mathbb{C}^m))$, for $\omega((S_t)_{t \geq 0})$ defined by (1.13) one has

$$\omega((S_t)_{t \geq 0}) = \inf \{ \omega \in \mathbb{R} : \{[e^{-\omega t} \exp(tA(\cdot))] \cdot : t \in [0, \infty[\} \text{ is an equicontinuous subset of } L_b(\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m); \mathcal{S}(\mathbb{R}^n; \mathbb{C}^m)) \}.$$

From [K3, Theorem 3.1] it follows that whenever $s_0 \in \mathbb{R}$, then

$$\omega((S_t)_{t \geq 0}) < s_0 + \varepsilon \quad \text{for every } \varepsilon > 0 \quad (4.1)$$

if and only if (2.8)* holds. Hence, by Proposition 2.3, the condition (4.1) is equivalent to (2.6). This implies that the growth bound of the i.d.c.s. $(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ is equal to the spectral bound of G , where both these quantities may well be infinite.

5 Condition (1.22) implies Gårding hyperbolicity

Let $(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ be an i.d.c.s. with generating distribution $G = \mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta$, so that the condition (1.14) is satisfied. Then $\mathcal{F}S_t \in \mathcal{O}_M(\mathbb{R}^n; M_{m \times m})$ and $(\mathcal{F}S_t)(\xi) = \exp(t\mathcal{G}(i\xi))$ for every $t \in [0, \infty[$ and $\xi \in \mathbb{R}^n$. Suppose that (1.22) holds, i.e. $S_{t_0} \in \mathcal{E}'(\mathbb{R}^n; M_{m \times m})$ for some $t_0 \in]0, \infty[$. Then, by the Paley–Wiener–Schwartz theorem, i.e. by [H, Theorem 7.3.1] or [K-R, Theorem 8.57], there are $C, k, l \in]0, \infty[$ such that whenever $\zeta \in \mathbb{C}^n$, then

$$\|\exp(t_0\mathcal{G}(i\zeta))\|_{M_{m \times m}} = \|(\mathcal{F}S_{t_0})(\zeta)\|_{M_{m \times m}} \leq C(1 + |\zeta|)^l e^{k \operatorname{Im} \zeta}. \quad (5.1)$$

For every $\zeta \in \mathbb{C}^n$ put

$$\Lambda(\zeta) = \max \operatorname{Re} \sigma(\mathcal{G}(i\zeta)).$$

Then

$$\Lambda(\zeta) = \max \{ \operatorname{Re} \lambda : \lambda \in \mathbb{C}, P(\lambda, \zeta_1, \dots, \zeta_n) = 0 \}$$

where

$$\begin{aligned} P(\lambda, \zeta_1, \dots, \zeta_n) &= \det(\lambda \mathbb{1}_{m \times m} - \mathcal{G}(\zeta_1, \dots, \zeta_n)) \\ &= \lambda^m + Q_{m-1}(\zeta_1, \dots, \zeta_n) \lambda^{m-1} \\ &\quad + \dots + Q_1(\zeta_1, \dots, \zeta_n) \lambda + Q_0(\zeta_1, \dots, \zeta_n). \end{aligned}$$

Let

$$p_0 = \inf \{ p \in]0, \infty[: \sup_{\zeta \in \mathbb{C}^n} (1 + |\zeta|)^{-p} \Lambda(\zeta) < \infty \}.$$

By (2.2) and (5.1) there is $K \in]0, \infty[$ such that

$$\Lambda(\zeta) \leq t_0^{-1} \log \|\exp(t_0\mathcal{G}(i\zeta))\|_{M_{m \times m}} \leq K(1 + |\zeta|)$$

for every $\zeta \in \mathbb{C}^n$. Consequently,

$$p_0 \leq 1. \quad (5.2)$$

By the Gelfand–Shilov theorem on the reduced order [G-S, Sec. II.6.2], [F, Sec. 7.2, Theorem 4],

$$p_0 = \max_{k=0, \dots, m-1} (m-k)^{-1} \deg Q_k,$$

so that, by (5.2), $\deg Q_k \leq m-k$ for every $k = 0, \dots, m-1$, and hence $\deg P = m$, proving (1.23).

6 Gårding hyperbolicity implies Ehrenpreis hyperbolicity

Suppose that $(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ is an i.d.c.s. with generating distribution $G = \mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta$. Let $P(\lambda, \zeta_1, \dots, \zeta_n) = \det(\lambda \mathbb{1}_{m \times m} - \mathcal{G}(\zeta_1, \dots, \zeta_n))$. Then, by Theorem 3, (1.14)' holds, i.e. $\sup\{\operatorname{Re} \lambda : \lambda \in \mathbb{C} \text{ and there is } (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \text{ such that } P(\lambda, i\xi_1, \dots, i\xi_n) = 0\} = s_0(\mathcal{G}) < \infty$. Suppose moreover that (1.23) holds, i.e. $\deg P(\lambda, \zeta_1, \dots, \zeta_n) = m$.

By [H, Theorem 12.4.2 and Lemma 8.7.3], the above properties of $P(\lambda, \zeta_1, \dots, \zeta_n)$ imply that Γ defined in our Sec. 1.5 is an open convex cone with vertex at zero. From the definition of Γ it follows that Γ contains the open halfline $\{(t, 0, \dots, 0) \in \mathbb{R}^{1+n} : t > 0\}$. From [H, Theorem 12.4.4] *) it follows that

whenever $(\nu_0, \nu_1, \dots, \nu_n) \in \Gamma$, $(\xi, \dots, \xi_n) \in \mathbb{R}^n$, $\lambda, \mu \in \mathbb{C}$, $\operatorname{Re} \lambda > s_0(\mathcal{G})$ and $\operatorname{Re} \mu \geq 0$, then

$$P(\lambda + \mu\nu_0, i\xi_1 + \mu\nu_1, \dots, i\xi_n + \mu\nu_n) \neq 0. \quad (6.1)$$

Fix $r > 0$ so large that

$$K_r := \{(\nu_0, \nu_1, \dots, \nu_n) \in \mathbb{R}^{1+n} : \nu_0 \geq r, \nu_1^2 + \dots + \nu_n^2 \leq 1\} \subset \Gamma.$$

Let $(\xi_1, \dots, \xi_n), (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$, $\mu = 1 + |\eta| = 1 + (\eta_1^2 + \dots + \eta_n^2)^{1/2}$, $(\nu_0, \nu_1, \dots, \nu_n) = (r, \eta_1/(1 + |\eta|), \dots, \eta_n/(1 + |\eta|))$. Then $(\nu_0, \dots, \nu_n) \in K_r \subset \Gamma$, and if $\lambda \in \mathbb{C}$ and

$$\operatorname{Re} \lambda > s_0(\mathcal{G}) + (1 + |\eta|)r,$$

then, by (6.1),

$$\begin{aligned} P(\lambda, i\xi_1 + \eta_1, \dots, i\xi_n + \eta_n) \\ = P((\lambda - (1 + |\eta|)r) + \mu\nu_0, i\xi_1 + \mu\nu_1, \dots, i\xi_n + \mu\nu_n) \neq 0 \end{aligned}$$

because $\operatorname{Re}(\lambda - (1 + |\eta|)r) > s_0(\mathcal{G})$. It follows that

whenever $(\lambda, \zeta_1, \dots, \zeta_n) \in \mathbb{C}^{1+n}$ and $P(\lambda, \zeta_1, \dots, \zeta_n) = 0$, then

$$\operatorname{Re} \lambda \leq s_0(\mathcal{G}) + r + r((\operatorname{Re} \zeta_1)^2 + \dots + (\operatorname{Re} \zeta_n)^2)^{1/2}. \quad (6.2)_+$$

By [G, Lemma 2.2] or [H, Theorem 12.4.1], if the polynomial $P(\lambda, \zeta_1, \dots, \zeta_n)$ satisfies (1.14)' and (1.23), then so does $P(-\lambda, \zeta_1, \dots, \zeta_n)$. Since $(6.2)_+$ is a

*) One could also use [G, Lemma 2.6], but in [G] the open convex cone Γ has a definition equivalent to but formally different from ours, which is taken from [H].

consequence of the properties (1.14)' and (1.23) of $P(\lambda, \zeta_1, \dots, \zeta_n)$, it follows that the properties (1.14)' and (1.23) of $P(-\lambda, \zeta_1, \dots, \zeta_n)$ imply that there is $r' > 0$ such that

whenever $(\lambda, \zeta_1, \dots, \zeta_n) \in \mathbb{C}^{1+n}$ and $P(\lambda, \zeta_1, \dots, \zeta_n) = 0$,
then

$$-\operatorname{Re} \lambda \leq s_0(-\mathcal{G}) + r' + r'((\operatorname{Re} \zeta_1)^2 + \dots + (\operatorname{Re} \zeta_n)^2)^{1/2}. \quad (6.2)_-$$

Together (6.2)₊ and (6.2)₋ mean that (1.27) is satisfied, i.e. the matricial PDO (1.26) is hyperbolic in the sense of Ehrenpreis with respect to the coordinate t .

7 The Ehrenpreis hyperbolicity implies (1.24)

Suppose that the system (1.26) is hyperbolic in the sense of Ehrenpreis with respect to the coordinate t . This means that whenever $(\lambda, \zeta_1, \dots, \zeta_n) \in \mathbb{C}^{1+n}$ and

$$P(\lambda, \zeta_1, \dots, \zeta_n) = \det(\lambda \mathbb{1}_{m \times m} - \mathcal{G}(\zeta_1, \dots, \zeta_n)) = 0,$$

then

$$|\operatorname{Re} \lambda| \leq C(1 + ((\operatorname{Re} \zeta_1)^2 + \dots + (\operatorname{Re} \zeta_n)^2)^{1/2})$$

for some $C \in]0, \infty[$ independent of $(\lambda, \zeta_1, \dots, \zeta_n)$. Since

$$\sigma(\mathcal{G}(i\zeta)) = \{\lambda \in \mathbb{C} : P(\lambda, i\zeta, \dots, i\zeta) = 0\},$$

it follows that whenever $(\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n$, then

$$\max |\operatorname{Re} \sigma(\mathcal{G}(i\zeta))| \leq C(1 + ((\operatorname{Im} \zeta_1)^2 + \dots + (\operatorname{Im} \zeta_n)^2)^{1/2}). \quad (7.1)$$

By (2.1) and (2.2), this implies that

$$\begin{aligned} \|\exp(t\mathcal{G}(i\zeta))\|_{M_{m \times m}} &\leq e^{C|t|} \left(1 + \sum_{k=1}^{m-1} \frac{(2|t|)^k}{k!} \|\mathcal{G}(i\zeta)\|_{M_{m \times m}}^k \right) e^{C|t| |\operatorname{Im} \zeta|} \\ &\leq e^{C|t|} (1 + 2|t|)^{m-1} D(1 + |\zeta|)^{(m-1)d} e^{C|t| |\operatorname{Im} \zeta|}, \end{aligned} \quad (7.2)$$

for every $(t, \zeta) \in \mathbb{R} \times \mathbb{C}$ where $C, D \in]0, \infty[$ are independent of (t, ζ) , and $d \in \mathbb{N}_0$ is the maximum of the orders of the scalar PDO which are the entries of $\mathcal{G}(\partial_1, \dots, \partial_n)$. By the Paley–Wiener–Schwartz theorem, i.e. by [H, Theorem 7.3.1], (7.2) implies that there is a one-parameter convolution group $(\tilde{S}_t)_{t \in \mathbb{R}} \subset \mathcal{E}'(\mathbb{R}^n; M_{m \times m})$ such that

$$(\mathcal{F}\tilde{S}_t)(\zeta) = \exp(t\mathcal{G}(i\zeta)) \quad \text{for every } (t, \zeta) \in \mathbb{R} \times \mathbb{C}^n \quad (7.3)$$

and

$$\max\{|x| : x \in \text{supp } \tilde{S}_t\} \leq C|t| \quad \text{for every } t \in \mathbb{R}.$$

The convolution group $(\tilde{S}_t)_{t \in \mathbb{R}}$ is an extension of the i.d.c.s. $(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ with generating distribution $\mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta$ which exists by Theorem 3 because (7.1) \Rightarrow (1.14). Furthermore, by (7.2), one has

$$\begin{aligned} \|\mathcal{G}(i\zeta)^k \exp(t\mathcal{G}(i\zeta))\|_{M_{m \times m}} \\ \leq e^{C|t|} (1 + 2|t|)^{m-1} D_k (1 + |\zeta|)^{(m+k-1)d} e^{C|t||\text{Im } \zeta|} \end{aligned} \quad (7.4)$$

for every $(t, \zeta) \in \mathbb{R} \times \mathbb{C}^n$ and $k \in \mathbb{N}_0$. By the theorem on differentiating a solution of an ODE with respect to a parameter ([Ha, Sec. V.4, Corollary 4.1]), the mapping $\mathbb{R} \times \mathbb{C}^n \ni (t, \zeta) \mapsto \exp(t\mathcal{G}(i\zeta)) \in M_{m \times m}$ is infinitely differentiable. Since $(\partial/\partial t)^k \exp(t\mathcal{G}(i\zeta)) = \mathcal{G}(i\zeta)^k \exp(t\mathcal{G}(i\zeta))$, from (7.4) and [E2, Sec. V.5, Lemma 5.17] it follows that the mapping $\mathbb{R} \ni t \mapsto \tilde{S}_t \in \mathcal{E}'(\mathbb{R}^n; M_{m \times m})$ is infinitely differentiable in the topology of $\mathcal{E}'(\mathbb{R}^n; M_{m \times m})$.

8 Application to the Cauchy problem

8.1 Well posedness spaces

Let $\mathcal{G}(\partial_1, \dots, \partial_n)$ be an $m \times m$ matrix whose entries are PDOs on \mathbb{R}^n with constant complex coefficients. Suppose that

$$\sup\{\text{Re } \lambda : \lambda \in \sigma(\mathcal{G}(i\xi)), \xi \in \mathbb{R}^n\} < \infty. \quad (\text{ii})$$

Then, by Theorem 3, there is a unique infinitely differentiable convolution semigroup $(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ with generating distribution $\mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta$. Suppose moreover that

$$\begin{aligned} E \text{ is a sequentially complete l.c.v.s. continuously imbedded in } \\ \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m) \text{ such that } (S_t *)E \subset E \text{ for every } t \in [0, \infty[, \text{ and} \\ \text{the mapping } [0, \infty[\times E \ni (t, u) \mapsto S_t * u \in E \text{ is separately} \\ \text{continuous.} \end{aligned} \quad (\text{iii})_{S_t, E}$$

Define the operator \mathcal{G}_E from E into E by the conditions

$$\begin{aligned} D(\mathcal{G}_E) &= \{u \in E : \mathcal{G}(\partial_1, \dots, \partial_n)u \in E\}, \\ \mathcal{G}_E u &= \mathcal{G}(\partial_1, \dots, \partial_n)u \quad \text{for } u \in D(\mathcal{G}_E). \end{aligned}$$

Theorem 5. *Suppose that conditions (ii) and (iii)_{S_t, E} are satisfied. Then for every $k = 1, 2, \dots, \infty$ the Cauchy problem*

$$\frac{d}{dt}u(t) = \mathcal{G}(\partial_1, \dots, \partial_n)u(t) \quad \text{for } t \in [0, \infty[, \quad u(0) = u_0, \quad (\text{iv})$$

with given $u_0 \in D(\mathcal{G}_E^k)$ has a solution $u(\cdot) \in C^k([0, \infty[; E)$ which is unique in the class $C^1([0, \infty[; \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m))$. This solution is given by the formula

$$u(t) = S_t * u_0 \quad \text{for } t \in [0, \infty[. \quad (\text{v})$$

Thanks to Theorem 5 it is legitimate to call E the *well posedness space* for the Cauchy problem (iv) if conditions (ii) and (iii) $_{S_t, E}$ are satisfied. Theorem 5 confirms the observation of L. Hörmander [H, notes at the end of Chapter 12] that the Petrovskii condition (ii) is related to well posedness of the Cauchy problem for PDOs with constant coefficients in L. Schwartz spaces \mathcal{S} and \mathcal{S}' .

Remark. Let $Z(\mathbb{C}^n; \mathbb{C}^m)$ be the space of \mathbb{C}^m -valued functions holomorphic on \mathbb{C}^n such that $\varphi \in Z(\mathbb{C}^n; \mathbb{C}^m)$ if and only if there is $a = a(\varphi) \in]0, \infty[$ such that $\sup_{z \in \mathbb{C}^n} (1 + \|z\|)^k e^{-a\|\text{Im } z\|} \|\varphi(z)\| < \infty$ for every $k \in \mathbb{N}$. Let $Z(\mathbb{C}^n; \mathbb{C}^m)|_{\mathbb{R}^n}$ be the set of restrictions to \mathbb{R}^n of functions in $Z(\mathbb{C}^n; \mathbb{C}^m)$. By the Paley–Wiener theorem ([H, Theorem 7.3.1], [K-R, Theorem 8.51]), $Z(\mathbb{C}^n; \mathbb{C}^m)|_{\mathbb{R}^n} = \mathcal{FD}(\mathbb{R}^n; \mathbb{C}^m)$, so that $Z(\mathbb{C}^n; \mathbb{C}^m)|_{\mathbb{R}^n}$ is a dense subset of $\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m) = \mathcal{FS}(\mathbb{R}^n; \mathbb{C}^m)$. If $E = Z(\mathbb{C}^n; \mathbb{C}^m)|_{\mathbb{R}^n}$, then the Cauchy problem (iv) is well posed for every $\mathcal{G}(\partial_1, \dots, \partial_n)$ independently of whether (ii) holds or not. Indeed, if $E = Z(\mathbb{C}^n; \mathbb{C}^m)|_{\mathbb{R}^n}$, then (instead of appealing to Sec. 2 which enables the use of the Lemma from Sec. 1.4) in order to conclude that the Cauchy problem (iv) is well posed it is sufficient to observe that the mapping $\mathbb{R} \ni t \mapsto [\exp t\mathcal{G}(i \cdot, \dots, i \cdot)] \cdot \in L_b(\mathcal{D}(\mathbb{R}^n; \mathbb{C}^m); \mathcal{D}(\mathbb{R}^n; \mathbb{C}^m))$ is infinitely differentiable because so is the mapping $\mathbb{R}^{1+n} \ni (t, \xi_1, \dots, \xi_n) \mapsto \exp(t\mathcal{G}(i\xi_1, \dots, i\xi_n)) \in M_{m \times m}$.

8.2 Examples of well posedness spaces

Examples of spaces E satisfying (iii) $_{S_t, E}$ for each $\mathcal{G}(\partial_1, \dots, \partial_n)$ satisfying (ii) include:

- (a) the spaces of infinitely differentiable functions $\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m)$ and $\mathcal{D}_{L^p}(\mathbb{R}^n; \mathbb{C}^m) = \{u \in C^\infty(\mathbb{R}^n; \mathbb{C}^m) : \partial^\alpha u \in L^p(\mathbb{R}^n; \mathbb{C}^m) \text{ for every } \alpha \in \mathbb{N}_0^n\}$, $p \in [1, \infty]$,
- (b) the spaces of distributions $\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$, $\mathcal{O}'_C(\mathbb{R}^n; \mathbb{C}^m)$ and

$$\mathcal{D}'_{L^q}(\mathbb{R}^n; \mathbb{C}^m) = (\mathcal{D}_{L^p})'(\mathbb{R}^n; \mathbb{C}^m), \quad q \in [1, \infty], \quad p = q/(q-1).$$

Examples of spaces E depending on $\mathcal{G}(\partial_1, \dots, \partial_n)$ such that the Cauchy problem (iv) is well posed whenever $\mathcal{G}(\partial_1, \dots, \partial_n)$ satisfies (ii) include:

(c) the T. Ushijima space

$$U_{\mathcal{G}}(\mathbb{R}^n; \mathbb{C}^m) = \{u \in L^2(\mathbb{R}^n; \mathbb{C}^m) : (\mathcal{G}(\partial_1, \dots, \partial_n))^k u \in L^2(\mathbb{R}^n; \mathbb{C}^m) \text{ for every } k = 1, 2, \dots\}$$

occurring in [U, Theorem 10.1],

(d) the Banach spaces $\mathcal{B}_{\mathcal{N},p}$ of G. Birkhoff [B],

(e) the Hilbert spaces \mathcal{L}_B of S. D. Eidelman and S. G. Krein discussed in [K, Sec. I.8.2].

In the cases (c)–(e) the well posedness of the Cauchy problem (iv) follows directly from the results of [U], [B] and [K] without reference to $\mathcal{O}'_C(\mathbb{R}^n; \mathbb{C}^m)$ and (v). Notice that in [P], [U] and [K2] it is proved that if E is equal to either of the spaces $\mathcal{D}_{L^\infty}(\mathbb{R}^n; \mathbb{C}^m)$, $\mathcal{D}_{L^2}(\mathbb{R}^n; \mathbb{C}^m)$ or $U_{\mathcal{G}}(\mathbb{R}^n; \mathbb{C}^m)$, then (ii) is necessary for well posedness of the Cauchy problem (iv). (The arguments from [P] and [U] are quoted in [K2].)

From among the spaces $\mathcal{D}_{L^p}(\mathbb{R}^n; \mathbb{C}^m)$, $p \in [1, \infty]$, the largest one is $\mathcal{D}_{L^\infty}(\mathbb{R}^n; \mathbb{C}^m)$ whose dual is not a space of distributions and does not occur in (b). The fact that if (ii) holds, then the Cauchy problem (iv) is well posed for $E = \mathcal{D}_{L^\infty}(\mathbb{R}^n; \mathbb{C}^m)$, was first proved by I. G. Petrovskii [P] in 1938. From among the spaces $\mathcal{D}'_{L^q}(\mathbb{R}^n; \mathbb{C}^m)$, $q \in]1, \infty]$, the largest one is $\mathcal{D}'_{L^\infty}(\mathbb{R}^n; \mathbb{C}^m)$, i.e. the space of \mathbb{C}^m -valued distributions on \mathbb{R}^n bounded in the sense of L. Schwartz. An alternative notation for \mathcal{D}'_{L^∞} is \mathcal{B}' .

The space $\mathcal{O}_M(\mathbb{R}^n; \mathbb{C}^m)$ cannot be included in (a) because $\mathcal{O}_M(\mathbb{R}^n; \mathbb{C}^1)$ is not a well posedness space when $\mathcal{G}(\partial_1, \dots, \partial_n) = i\Delta$. Indeed $i\Delta$ satisfies (ii) and, in accordance with Example 1 of Sec. 1.4, the i.d.c.s. whose generating distribution is $i\Delta\delta$ extends to a one-parameter group $(S_t)_{t \in \mathbb{R}} \subset \mathcal{O}'_C(\mathbb{R}^n)$ such that $S_t \in \mathcal{O}'_C(\mathbb{R}^n) \cap \mathcal{O}_M(\mathbb{R}^n)$ for every $t \in \mathbb{R} \setminus \{0\}$. Fix $t_0 \in]0, \infty[$. Then, by Theorem 5, the Cauchy problem

$$\frac{d}{dt}u(t) = i\Delta u(t) \quad \text{for } t \in [0, \infty[, \quad u(0) = S_{-t_0}, \quad (\text{iii})_0$$

has in the class $C^1([0, \infty[; \mathcal{S}'(\mathbb{R}^n))$ a unique solution. Since this unique solution is given by the formula $u(t) = S_t * S_{-t_0}$ it follows that $u(0) = S_{-t_0} \in \mathcal{O}_M(\mathbb{R}^n)$ and $u(t_0) = \delta \notin \mathcal{O}_M(\mathbb{R}^n)$. Consequently, the Cauchy problem (iii)₀ has no solution in the class $C^1([0, \infty[; \mathcal{O}_M(\mathbb{R}^n))$.

8.3 Well posedness of the spaces $\mathcal{D}_{L^p}(\mathbb{R}^n; \mathbb{C}^m)$

We shall use the following

Lemma. *Let $(S_t)_{t \geq 0} \subset \mathcal{O}_C(\mathbb{R}^n; M_{m \times m})$ be an i.d.c.s. with generating distribution $\mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta$ satisfying (ii). Then there are $j_0 \in \mathbb{N}$ and a*

continuous mapping $[0, \infty[\ni t \mapsto f_t \in L^1(\mathbb{R}^n; M_{m \times m})$ having the three properties:

- (a) $f_t \in L^1(\mathbb{R}^n; M_{m \times m}) \cap \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ for every $t \in [0, \infty[$,
- (b) $S_t = (1 - \Delta)^{j_0} f_t$ for every $t \in [0, \infty[$ where the right side is understood in the sense of $\mathcal{S}'(\mathbb{R}^n; M_{m \times m})$,
- (c) $\sup_{t \in [0, \infty[} e^{-(s_0(\mathcal{G}) + \varepsilon)t} \|f_t\|_{L^1(\mathbb{R}^n; M_{m \times m})} < \infty$ for every $\varepsilon > 0$.

Before proving the lemma let us show how it implies that $\mathcal{D}_{L^p}(\mathbb{R}^n; \mathbb{C}^m)$, $p \in [1, \infty]$, are well posedness spaces. Let $(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ be an i.d.c.s. whose generating distribution $\mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta$ satisfies (ii). Whenever $t \in [0, \infty[$ and $u \in \mathcal{D}_{L^p}(\mathbb{R}^n; \mathbb{C}^m)$ then

$$S_t * u = ((1 - \Delta)^{j_0} f_t) * u = f_t * ((1 - \Delta)^{j_0} u)$$

because $S_t, f_t \in \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ and $u \in \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$. The continuity of the mapping $[0, \infty[\ni t \mapsto f_t \in L^1(\mathbb{R}^n; M_{m \times m})$ implies the separate continuity of the mapping

$$[0, \infty[\times \mathcal{D}_{L^p}(\mathbb{R}^n; \mathbb{C}^m) \ni (t, u) \mapsto S_t * u = f_t * ((1 - \Delta)^{j_0} u) \in \mathcal{D}_{L^p}(\mathbb{R}^n; \mathbb{C}^m).$$

Finally, $\mathcal{D}_{L^p}(\mathbb{R}^n; \mathbb{C}^m)$ is sequentially complete, and it is continuously imbedded in $\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$. Consequently, the condition (iii) $_{S_t, E}$ is satisfied for $E = \mathcal{D}_{L^p}(\mathbb{R}^n; \mathbb{C}^m)$.

Proof of the Lemma. By the estimation (2.8) from Proposition 2.3, and by the statement (2.8) from [K3], for every fixed $j_0 \in \mathbb{N}$ and $t \in [0, \infty[$ the function $g_t : \mathbb{R}^n \ni \xi \mapsto (1 + |\xi|^2)^{-j_0} \exp(t\mathcal{G}(i\xi)) \in M_{m \times m}$ belongs to $\mathcal{O}_M(\mathbb{R}^n; M_{m \times m})$. Let $f_t = \mathcal{F}^{-1}g_t$. Then $f_t \in \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ and $(1 - \Delta)^{j_0} f_t = \mathcal{F}^{-1}(\exp(t\mathcal{G}(i \cdot))) = S_t$. The Lemma follows once we prove that if j_0 is sufficiently large, then each distribution $f_t \in \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$, $t \in [0, \infty[$, is represented by a function belonging to $L^1(\mathbb{R}^n; M_{m \times m})$ such that the mapping $[0, \infty[\ni t \mapsto f_t \in L^1(\mathbb{R}^n; M_{m \times m})$ is locally lipschitzian and satisfies (c). We shall base on the fact that

if $T \in \mathcal{S}'(\mathbb{R}^n; M_{m \times m})$ and $(1 - \Delta)^{[n/2]+1} \hat{T} \in L^1(\mathbb{R}^n; M_{m \times m})$,
then $T \in L^1(\mathbb{R}^n; M_{m \times m})$ and

$$\|T\|_{L^1(\mathbb{R}^n; M_{m \times m})} \leq C \|(1 - \Delta)^{[n/2]+1} \hat{T}\|_{L^1(\mathbb{R}^n; M_{m \times m})} \quad (8.1)$$

where $C \in]0, \infty[$ depends only on n . To prove (8.1) it is sufficient to note that one has dense imbeddings $\mathcal{S}(\mathbb{R}^n; M_{m \times m}) \subset L^1(\mathbb{R}^n; M_{m \times m}) \subset \mathcal{S}'(\mathbb{R}^n; M_{m \times m})$ and if $\varphi \in \mathcal{S}(\mathbb{R}^n; M_{m \times m})$, then

$$\begin{aligned} \|\varphi\|_{L^1(\mathbb{R}^n; M_{m \times m})} &\leq C \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^{[n/2]+1} \|\varphi(x)\|_{M_{m \times m}} \\ &\leq C \|(1 - \Delta)^{[n/2]+1} \hat{\varphi}\|_{L^1(\mathbb{R}^n; M_{m \times m})}. \end{aligned}$$

In order to prove the Lemma, we shall apply (8.1) to $T = (d/dt)^l f_t$ where $l = 0, 1$ and $t \in [0, \infty[$. For this T one has

$$((1 - \Delta)^{[n/2]+1} \hat{T})(\xi) = (1 - \Delta_\xi)^{[n/2]+1} ((1 + |\xi|^2)^{-j_0} (\mathcal{G}(i\xi))^l \exp(t\mathcal{G}(i\xi))),$$

so that $(1 - \Delta)^{[n/2]+1} \hat{T} \in \mathcal{O}_M(\mathbb{R}^n; M_{m \times m})$, again by (2.8) from Proposition 2.3 and (2.8) from [K3]. In order to show that if j_0 is sufficiently large, then $(1 - \Delta)^{[n/2]+1} \hat{T} \in L^1(\mathbb{R}^n; M_{m \times m})$, it is sufficient to prove that whenever $j_0 \in \mathbb{N}$ is sufficiently large and $\kappa \in \mathbb{N}_0^n$ is a multiindex of length $|\kappa| \leq n + 2$, then the $M_{m \times m}$ -valued function

$$\xi \mapsto \left(\frac{\partial}{\partial \xi} \right)^\kappa [(1 + |\xi|^2)^{-j_0} (\mathcal{G}(i\xi))^l \exp(t\mathcal{G}(i\xi))]$$

is integrable on \mathbb{R}^n .

The Leibniz formula, the estimation (2.8) from Proposition 2.3, and the statement (2.8) from [K3] imply that for every $\varepsilon > 0$ there is $D_\varepsilon \in]0, \infty[$ such that whenever $|\kappa| \leq n + 2$, then

$$\begin{aligned} & \sum_{|\kappa| \leq n+2} \int_{\mathbb{R}^n} \left\| \left(\frac{\partial}{\partial \xi} \right)^\kappa [(1 + |\xi|^2)^{-j_0} (\mathcal{G}(i\xi))^l \exp(t\mathcal{G}(i\xi))] \right\|_{M_{m \times m}} d\xi \\ & \leq \sum_{|\alpha|+|\beta|+|\gamma| \leq n+2} \frac{(\alpha + \beta + \gamma)!}{\alpha! \beta! \gamma!} \int_{\mathbb{R}^n} \left| \left(\frac{\partial}{\partial \xi} \right)^\gamma (1 + |\xi|^2)^{-j_0} \right| \\ & \quad \cdot \left\| \left(\frac{\partial}{\partial \xi} \right)^\beta (\mathcal{G}(i\xi))^l \right\|_{M_{m \times m}} \cdot \left\| \left(\frac{\partial}{\partial \xi} \right)^\alpha \exp(t\mathcal{G}(i\xi)) \right\|_{M_{m \times m}} d\xi \\ & \leq D_\varepsilon e^{(s_0(\mathcal{G})+\varepsilon)t} \sum_{|\alpha|+|\beta|+|\gamma| \leq n+2} \frac{(\alpha + \beta + \gamma)!}{\alpha! \beta! \gamma!} \int_{\mathbb{R}^n} (1 + |\xi|^2)^{-j_0 - \frac{1}{2}|\gamma| + ld + k_\alpha} d\xi. \end{aligned}$$

In the above d is the maximum of the orders of the scalar PDOs which are entries of $\mathcal{G}(\partial_1, \dots, \partial_n)$. If j_0 is sufficiently large, then all the integrals in the last member of the estimate are finite, so that $(1 - \Delta)^{[n/2]+1} (d/dt)^l \hat{f}_t \in L^1(\mathbb{R}^n; M_{m \times m})$ and

$$\left\| (1 - \Delta)^{[n/2]+1} \left(\frac{d}{dt} \right)^l \hat{f}_t \right\|_{L^1(\mathbb{R}^n; M_{m \times m})} \leq K_\varepsilon e^{(s_0(\mathcal{G})+\varepsilon)t}$$

for every $t \in [0, \infty[$, $l = 0, 1$ and $\varepsilon > 0$, where $K_\varepsilon \in]0, \infty[$ is independent of t and l . By (8.1), this implies that $f_t \in L^1(\mathbb{R}^n; M_{m \times m})$ for every $t \in [0, \infty[$, and the mapping $[0, \infty[\ni t \mapsto f_t \in L^1(\mathbb{R}^n; M_{m \times m})$ is continuous and satisfies (c).

8.4 Well posedness of the dual spaces

Let E be an l.c.v.s. continuously imbedded in $\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$. Suppose moreover that $\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m)$ is densely and continuously imbedded in E . Let

$$E' = \left\{ T \in \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m) : \text{the linear functional} \right.$$

$$\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m) \ni \varphi \mapsto \langle T, \varphi \rangle = \sum_{\mu=1}^m T_\mu(\varphi_\mu) \in \mathbb{C}$$

is continuous on $\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m)$ in the topology induced by E $\left. \right\}$.

Then each $T \in E'$ uniquely extends to a continuous functional on E , and may be identified with that functional. E' is equipped with the topology of uniform convergence on bounded subsets of E .

Assume that $\mathcal{G}(\partial_1, \dots, \partial_n)$ satisfies (ii), and let $(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ be the i.d.c.s. with generating distribution $\mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta$. Then $\mathcal{G}^\dagger(-\partial_1, \dots, -\partial_n)$ also satisfies (ii), and $(\check{S}_t^\dagger)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$ is the i.d.c.s. with generating distribution $\mathcal{G}^\dagger(-\partial_1, \dots, -\partial_n) \otimes \delta$. (The above observation is related to [S1, Sec. 14].) Assume in addition that $\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m)$ is dense in the l.c.v.s. E , and E is a Montel and hence barrelled space continuously imbedded in $\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$. Then

$$(iii)_{\check{S}_t^\dagger, E} \quad \text{implies} \quad (iii)_{S_t, E'}.$$

Indeed, the sequential completeness of E' is a consequence of the barrelledness of E . Continuous imbedding of E' in $\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$ follows from dense and continuous imbedding of $\mathcal{S}(\mathbb{R}^n; \mathbb{C}^m)$ in E . The other properties of $(S_t)_{t \geq 0}$ and E' listed in $(iii)_{S_t, E'}$ follow from the equality

$$\langle S_t * T, u \rangle = \langle T, \check{S}_t^\dagger * u \rangle$$

for all $t \in [0, \infty[$, $u \in E$ and $T \in E'$.

8.5 Well posedness of $\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$ and uniqueness of solutions

If $\mathcal{G}(\partial_1, \dots, \partial_n)$ satisfies (ii), then, by the argument presented in Sec. 8.4, $\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$ is a well posedness space. Consequently,

$$((S_t *)|_{\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)})_{t \geq 0} \subset L(\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m); \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m))$$

is a one-parameter operator semigroup of class (C_0) . Since $\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$ is a Montel and hence barrelled space, from infinite differentiability of the

convolution semigroup $(S_t)_{t \geq 0} \subset \mathcal{O}'_C(\mathbb{R}^n; M_{m \times m})$, it follows by the Banach–Steinhaus theorem that the operator semigroup $((S_t *)|_{\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)})_{t \geq 0}$ is infinitely differentiable in the topology of $L_b(\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m); \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m))$. The infinitesimal generator of this operator semigroup is the differential operator

$$\mathcal{G}(\partial_1, \dots, \partial_n) \in L(\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m); \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)).$$

From the above it is easy to infer that

$$\begin{aligned} \frac{d}{dt}[(S_t *)|_{\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)}] &= \mathcal{G}(\partial_1, \dots, \partial_n)(S_t *)|_{\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)} \\ &= S_t * \mathcal{G}(\partial_1, \dots, \partial_n)|_{\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)} \end{aligned}$$

for every $t \in [0, \infty[$, where the derivative is computed in the topology of $L_b(\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m); \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m))$, and is understood as the two-sided derivative if $t \in]0, \infty[$, and as the right derivative if $t = 0$. Consequently, if $u_0 \in \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$, then the Cauchy problem (iv) has the solution $u(\cdot) \in C^\infty([0, \infty[; \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m))$ given by formula (v). If $\tilde{u}(\cdot) \in C^1([0, \infty[; \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m))$ is any other solution of (iv), then for every $t \in]0, \infty[$ and $\tau \in]0, t[$ one has

$$\begin{aligned} \frac{d}{d\tau}(S_{t-\tau} * \tilde{u}(\tau)) &= \lim_{]0, t-\tau] \ni h \rightarrow 0} \frac{1}{h}(S_{t-\tau-h} - S_{t-\tau}) * \tilde{u}(\tau) \\ &\quad + \lim_{]0, t-\tau] \ni h \rightarrow 0} S_{t-\tau-h} * \frac{d}{d\tau} \tilde{u}(\tau) \\ &\quad + \lim_{]0, t-\tau] \ni h \rightarrow 0} S_{t-\tau-h} * \left[\frac{1}{h}(\tilde{u}(\tau+h) - \tilde{u}(\tau)) - \frac{d}{d\tau} \tilde{u}(\tau) \right] \\ &= \left(\frac{d}{d\tau} S_{t-\tau} \right) * \tilde{u}(\tau) + S_{t-\tau} * \frac{d}{d\tau} \tilde{u}(\tau) \\ &= [S_{t-\tau} * (-\mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta)] * \tilde{u}(\tau) \\ &\quad + S_{t-\tau} * [(\mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta) * \tilde{u}(\tau)] = 0 \end{aligned}$$

in the topology of $\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$. Indeed, $\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$ is barrelled, so that, by the Banach–Steinhaus theorem, the set of convolution operators $\{S_{t-\tau-h} * : h \in [0, t-\tau]\}$ is an equicontinuous subset of $L(\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m); \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m))$, whence

$$\lim_{]0, t-\tau] \ni h \rightarrow 0} S_{t-\tau-h} * \left[\frac{1}{h} \left(\tilde{u}(\tau+h) - \tilde{u}(\tau) - \frac{d}{d\tau} \tilde{u}(\tau) \right) \right] = 0.$$

Consequently, the continuous function $[0, t] \ni \tau \mapsto S_{t-\tau} * \tilde{u}(\tau) \in \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$ is constant in the open interval $]0, t[$, and hence in $[0, t]$. It follows that $\tilde{u}(t) - S_t * u_0 = S_{t-\tau} * \tilde{u}(\tau)|_{\tau=0}^{\tau=t} = 0$. Hence in the class $C^1([0, \infty[; \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m))$ the Cauchy problem (iv) with $u_0 \in \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$ has a unique solution, and this solution belongs to $C^\infty([0, \infty[; \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m))$ and is represented by (v).

8.6 Proof of Theorem 5

Suppose that (ii) holds and E is any l.c.v.s. satisfying (iii) _{S_t, E} . Then $((S_t *)|_E)_{t \geq 0} \subset L(E, E)$ is a (C_0) -semigroup of operators. Fix $u_0 \in E$. The trajectory $t \mapsto S_t * u_0$ belongs to $C([0, \infty[; E)$ and, in view of Sec. 8.5, it belongs to $C^\infty([0, \infty[; \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m))$ and the right derivative $\frac{d}{dt}|_{t=0}(S_t * u_0)$ computed in the topology of $\mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m)$ is equal to $\mathcal{G}(\partial_1, \dots, \partial_n)u_0$. If the right derivative $\frac{d}{dt}|_{t=0}(S_t * u_0)$ exists in the topology of E , then it has the same value $\mathcal{G}(\partial_1, \dots, \partial_n)u_0$. This leads to the conclusion that the infinitesimal generator of the operator semigroup $((S_t *)|_E)_{t \geq 0} \subset L(E; E)$ is equal to the operator \mathcal{G}_E . By [K1, Theorem 3.3] (based on the R. E. Edwards boundedness principle for sequentially complete l.c.v.s. [E, Theorem 7.4.4]), whenever $u_0 \in D(\mathcal{G}_E)$ and $t \in]0, \infty[$, then $S_t * u_0 \in D(\mathcal{G}_E)$ and the two-sided derivative $\frac{d}{dt}(S_t * u_0)$ computed in the topology of E satisfies the equalities

$$\frac{d}{dt}(S_t * u_0) = \mathcal{G}_E(S_t * u_0) = S_t * (\mathcal{G}_E u_0). \quad (8.2)$$

Consequently, $u(t) = S_t * u_0$ belongs to $C_1([0, \infty[; E)$ and is a solution of the Cauchy problem (iv). The uniqueness of this solution is a consequence of Sec. 8.5 and the fact that $u(\cdot) \in C^\infty([0, \infty[; \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^m))$. From (8.2) it follows that if $u_0 \in D(\mathcal{G}_E^k)$ where $k = 2, 3, \dots$, then $u(\cdot) \in C^k([0, \infty[; E)$.

Appendix. Proof of (1.28)

Let $\mathcal{G}(\partial_1, \dots, \partial_n)$ be an $m \times m$ matrix whose entries are PDOs on \mathbb{R}^n with constant complex coefficients. Assume that conditions (1.14) and (1.23) are satisfied, so that, by Theorem 4, there is a unique i.d.c.g. $(S_t)_{t \in \mathbb{R}} \subset \mathcal{E}'(\mathbb{R}^n; M_{m \times m})$ with generating distribution $\mathcal{G}(\partial_1, \dots, \partial_n) \otimes \delta$. Take any $t_0 \in]0, \infty[$. By the Paley–Wiener–Schwartz theorem, i.e. by [H, Theorem 7.3.1] or [K-R, Theorem 8.57], $\mathcal{F}S_{t_0} = \exp(t_0 \mathcal{G}(i \cdot))$ can be extended to an $M_{m \times m}$ -valued function holomorphic on \mathbb{C}^n such that for some $C \in]0, \infty[$ and $l \in \mathbb{N}_0$ one has

$$\|\exp(t_0 \mathcal{G}(i\zeta))\|_{M_{m \times m}} = \|(\mathcal{F}S_{t_0})(\zeta)\|_{M_{m \times m}} \leq C(1 + |\zeta|)^l e^{H_0(\operatorname{Im} \zeta)}$$

for every $\zeta \in \mathbb{C}^n$ where $\mathbb{R}^n \ni \eta \mapsto H_0(\eta) = \sup\{x\eta : x \in \operatorname{supp} S_{t_0}\} \in \mathbb{R}$ is the supporting function of $\operatorname{supp} S_{t_0}$. By (2.1) and (2.2), it follows that

$$e^{t_0 \max \operatorname{Re} \sigma(\mathcal{G}(i\zeta))} \leq C(1 + |\zeta|)^l e^{H_0(\operatorname{Im} \zeta)},$$

and so

$$\begin{aligned}
\|(\mathcal{F}S_t)(\zeta)\|_{M_{m \times m}} &= \|\exp(t\mathcal{G}(i\zeta))\|_{M_{m \times m}} \\
&\leq (e^{t_0 \max \operatorname{Re} \sigma(\mathcal{G}(i\zeta))})^{t_0^{-1}t} \left(1 + \sum_{k=1}^{m-1} \frac{(2t)^k}{k!} \|\mathcal{G}(i\zeta)\|_{M_{m \times m}}^k \right) \\
&\leq C t_0^{-1} t (1 + |\zeta|)^{l t_0^{-1} t} e^{t_0^{-1} t H_0(\operatorname{Im} \zeta)} \left(1 + \sum_{k=1}^{m-1} \frac{(2t)^k}{k!} \|\mathcal{G}(i\zeta)\|_{M_{m \times m}}^k \right)
\end{aligned}$$

for every $(t, \zeta) \in [0, \infty[\times \mathbb{C}^n$. Since $\mathcal{G}(i\zeta)$ is an $m \times m$ matrix with polynomial entries, it follows that for every $t \in [0, \infty[$ there are $C_t, k_t \in]0, \infty[$ such that $\|(\mathcal{F}S_t)(\zeta)\|_{M_{m \times m}} \leq C_t (1 + |\zeta|)^{k_t} e^{t_0^{-1} t H_0(\operatorname{Im} \zeta)}$. Hence, by the Paley–Wiener–Schwartz theorem,

$$\frac{1}{t} \operatorname{conv} \operatorname{supp} S_t \subset \frac{1}{t_0} \operatorname{conv} \operatorname{supp} S_{t_0} \quad \text{for every } t \in [0, \infty[.$$

Since t and t_0 can be interchanged, one concludes that

$$\frac{1}{t} \operatorname{conv} \operatorname{supp} S_t = \frac{1}{t_0} \operatorname{conv} \operatorname{supp} S_{t_0} \quad \text{for every } t \in [0, \infty[. \quad (\text{A.1})$$

The formula

$$\tilde{E}(\varphi) = \int_0^\infty S_t(\varphi(t, \cdot)) dt, \quad \varphi \in \mathcal{D}(\mathbb{R}^{1+n}),$$

defines a fundamental solution \tilde{E} for the matricial PDO (1.26) with support contained in the cone

$$\begin{aligned}
K &= \{(t, x_1, \dots, x_n) \in \mathbb{R}^{1+n} : t \geq 0, (x_1, \dots, x_n) \in \operatorname{conv} \operatorname{supp} S_t\} \\
&= \{(t, x_1, \dots, x_n) \in \mathbb{R}^{1+n} : t \geq 0, (x_1, \dots, x_n) \in t t_0^{-1} \operatorname{conv} \operatorname{supp} S_{t_0}\}
\end{aligned}$$

where the equality is a consequence of (A.1). Consequently, $E = \det_* \tilde{E}$ is a fundamental solution for the operator $P(\partial_1, \partial_1, \dots, \partial_n) = \det(\mathbb{1}_{m \times m} \otimes \partial_t - \mathcal{G}(\partial_1, \dots, \partial_n))$. In the above \det_* is the determinant in the sense of the convolution algebra $\mathcal{O}'_C(\mathbb{R}^n)$. It follows that $\operatorname{supp} E \subset K$. Since $K \subset H_+ := \{(t, x_1, \dots, x_n) \in \mathbb{R}^{1+n} : t \geq 0\}$, from [H, Theorem 12.5.1] it follows that

$$\operatorname{supp} E \subset \Gamma^0, \quad (\text{A.2})$$

and

$$\text{whenever } H \text{ is a convex cone with } \operatorname{supp} E \subset H \subset H_+, \text{ then } \Gamma^0 \subset H. \quad (\text{A.3})$$

From (A.3) it follows that

$$\Gamma^0 \subset K. \quad (\text{A.4})$$

In order to prove the inclusion opposite to (A.4), first we shall show that

$$\text{supp } \tilde{E} \subset \Gamma^0. \quad (\text{A.5})$$

The above inclusion follows from (A.2) and the equality

$$\tilde{E} = [\text{adj}(\mathbb{1}_{m \times m} \otimes \partial_t - \mathcal{G}(\partial_1, \dots, \partial_n))] \otimes E. \quad (\text{A.6})$$

Indeed, $E_1 = \tilde{E}$ and $E_2 = [\text{adj}(\mathbb{1}_{m \times m} \otimes \partial_t - \mathcal{G}(\partial_1, \dots, \partial_n))] \otimes E$ both have support contained in H_+ , and both are fundamental solutions for the matricial PDO (1.26). Moreover $\vartheta_0 E_i \in \mathcal{E}'(\mathbb{R}^{1+n}; M_{m \times m})$ for $i = 1, 2$ and every $\vartheta_0 \in C^\infty(\mathbb{R}^{1+n})$ such that $\vartheta_0(t, x_1, \dots, x_n) \equiv \vartheta(t)$ where $\vartheta \in \mathcal{D}(\mathbb{R})$. These properties of E_i , $i = 1, 2$, imply the equality $E_1 = E_2$ (see the author's preprint *The Petrovskiĭ condition and rapidly decreasing distributions*, Inst. Math., Polish Acad. Sci., 2011). The equality $E_1 = E_2$ means that (A.6) holds. Now, (A.5) is a consequence of (A.2) and (A.6).

From (A.5) the inclusion

$$K \subset \Gamma^0 \quad (\text{A.7})$$

may be deduced by an elementary reasoning. Indeed, (A.7) follows once it is proved that

$$\text{supp } S_t \subset \Gamma_t^0 := \{(x_1, \dots, x_n) \in \mathbb{R}^n : (t, x_1, \dots, x_n) \in \Gamma^0\} \quad (\text{A.8})$$

for every $t \in [0, \infty[$. If (A.8) were not true for some $t_0 \in [0, \infty[$, then there would exist $\varphi_0 \in \mathcal{D}(\mathbb{R}^n)$ such that $(\text{supp } \varphi_0) \cap \Gamma_{t_0}^0 = \emptyset$ and $0 \neq S_{t_0}(\varphi_0) \in M_{m \times m}$. Since $S_t(\varphi_0)$ depends continuously on t , there would exist $\psi_0 \in \mathcal{D}(]0, \infty[)$ such that $(\text{supp}(\psi_0 \otimes \varphi_0)) \cap \Gamma^0 = \emptyset$ and $\tilde{E}(\psi_0 \otimes \varphi_0) = \int_0^\infty \psi_0(t) S_t(\varphi_0) dt \neq 0$, contrary to (A.5). Therefore (A.8) is true, and (A.7) holds. The inclusions (A.4) and (A.7) prove the equality (1.28).

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